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Engineering of Learning: Conceptualizing e-Didactics

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Mourat Tchoshanov

Engineering of Learning:

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Moscow
2013

UNESCO Institute for Information Technologies in Education

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Reviewer: Marina Tsvetkova, Academy of Teacher Professional Development
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The digital age demands re-thinking of traditional teaching and learning. Rapidly growing technological innovations in education force a paradigm shift from traditional teaching to engineering of learning. The main focus of the book is on the design, development, and implementation of effective learning environments through the use of Information and Communication Technologies in various formats: face-to-face, blended, and distance education. Engineering of learning requires new understanding and reconceptualization of traditional didactics toward e-Didactics in order to effectively design and skillfully align learning objectives, content, and assessment in the digital age classroom.

Published by the UNESCO Institute for Information Technologies in Education
8 Kedrova St., Bldg. 3, Moscow, 117292, Russian Federation
Tel.: +7 499 1292990
Fax: +7 499 1291225
E-mail: Liste.info.iite@unesco.org
www.iite.unesco.org

ISBN 978-5-905385-14-8
Printed in the Russian Federation

Contents

Preface 5
Introduction. From Teaching to Engineering of Learning 7

Chapter 1. **e-Didactics: Digital Age Didactics** 13
1.1. The Origins of Didactics 15
1.2. Didactical Triangle and Didactical Tetrahedron 18
1.3. e-Didactics and Didactical Engineering 21

Chapter 2. **Learning Sciences in the Digital Age** 31
2.1. The Guiding Principles of Learning 33
2.2. Constructivism 41
2.3. Constructionism 45
2.4. Social Constructivism in Action 49
2.5. Learning Culture and Multiple Intelligences 56

Chapter 3. **The Engineering of Learning Toolkit** 61
3.1. Design of Learning Objectives, Tasks, and Didactical Situations 63
3.2. Cognitive Tutoring, Representations, and New Literacies 71
3.3. Research-Based Strategies in Engineering of Learning 80
3.4. Assessment of Learning Outcomes 85

Chapter 4. **Engineering of Content** 91
4.1. Modular Design and Content Development 93
4.2. Content Interactivity and Content Communication 112
4.3. Engineering of Distance Learning 125

Conclusion 139
References 141
Websites 156
Appendix 157
Recent IITE publications 187

Preface

Following the development of innovative pedagogical technologies, the common apprehension of pedagogy and didactics as "the art and science of teaching and instructional theory" evolves together with the development of learning theories. In the digital age, traditional learning theories – behaviorism, constructionism and cognitivism are supplemented by new approaches, for example, connectivism, which suggests learning in the process of communication and connection within a distributed network and can be exemplified by the actively spreading Massive Open Online Courses.

The 21st-century teacher is equipped with numerous ICT tools and digital content to enhance or change students' experience of education; however, the fact that a teacher uses digital materials or electronic tools does not necessarily suggest that s/he is practicing appropriate pedagogical approaches tailored for the new tools. Pre-service and in-service training of teachers still provides an insight mainly into the traditional pedagogy; however, though digital pedagogy is emerging and some teachers experiment with new approaches, it still has to become a common ground. The digital age demands an adequate revision of pedagogical approaches. To meet the needs of contemporary students new ways of ICT-enhanced teaching and learning should be developed. The reframing and reconceptualization of traditional didactics, pedagogies and the learning landscape should become a prerequisite for a more efficient use of ICT in primary, secondary and higher schools. Digital pedagogies should be designed in accordance with the following principles: *authentic personalized learning, broadening experience and deepening knowledge, and learning in the global context* (ACCE, 2008).

UNESCO recognizes the need to refocus thinking about the use of ICT in education, shift the focus from the ICT tools to learning needs and pedagogies, novel approaches to using the new tools. This publication of the UNESCO Institute for Information Technologies in Education covers various aspects related to the concept of e-Didactics, from the origins of didactics to didactical engineering. The main focus of the book is on the design, development, and implementation of effective learning environments through the use of Information and Communication Technologies in various formats: face-to-face, blended, and distance education. The author, Dr. Mourat Tchoshanov, considers advantages and disadvantages of various learning theories and their modifications, for example, social constructivism in action. He analyses the engineering of learning from the viewpoint of a learning toolkit, which includes the design of learning objectives, tasks, and didactical situations, cognitive tutoring, assessment of learning outcomes, etc. The chapter devoted to the engineering of content outlines modular design and content development, content interactivity and communication, as well as engineering of distance learning. The book concludes by an example of a unit developed by the Texas-Science, Technology, Engineering, and Mathematics team, which should be of help for all those willing to master the new didactic approaches.

I hope that this book will become a helpful tool for educators at different levels and sectors of education in their transition to e-Didactics.

Dendev Badarch
IITE UNESCO Director a.i.

Introduction

From Teaching to Engineering of Learning

Since 2000 the author has been studying the approaches to the use of Information and Communication Technologies (ICT) in education and distance learning. In 2001, he developed an open access web site "Visual Mathematics" (http://mourat.utep.edu/vis_math/visuala.html) and used dynamic cognitive visualization to represent solutions to mathematical problems and proofs. The website is used by the author in mathematics methods and mathematics classes at the University of Texas at El Paso, USA.

During the recent years the author has been developing and teaching hybrid/blended (partially online) and distance (online) courses for pre-service and in-service training of secondary school teachers of mathematics. The analysis, modeling and designing of distance learning courses convinced the author that content and didactical knowledge are necessary but not sufficient for the development of high-quality online courses. In addition, one needs to acquire a new type of knowledge that integrates content, didactics, and engineering. Application of engineering approaches to didactics is called *didactical engineering*.

In addition to teaching online, the author's enthusiasm about the efficiency of didactical engineering was supported by working with mathematics teachers in an urban public school in the southwest Texas attended by about 750 students. The school was equipped with computer labs; each mathematics classroom was equipped with a computer for teachers and a few (three to five) computers for students to work on individual tasks and projects. Each classroom was also equipped with a projector and an interactive whiteboard. Mathematics classes were block-scheduled each day for 90 minutes. The Department of Mathematics employed 11 teachers whose teaching experience varied from one to 25 years.

According to the results of the state standardized test, in 2003-2005 the average level of achievement in mathematics of school students was around 41-46%. Teachers explained the low rate by students' reluctance to learn. During summer 2005, a group of teachers invited the author to work together with the Department of Mathematics to improve academic performance of students. The analysis of the curriculum, interviews with teachers, students, and parents revealed that poor performance was due to teachers rather than students.

After a thorough analysis of the situation the author proposed to improve performance using the didactical engineering approach. The hypothesis was that the poor student performance was stipulated by the attitude of a teacher, his/ her teaching methodology and subject matter proficiency. 'The vicious circle' had to be broken, passive teacher behavior (as teacher-technician) had to be changed to an active one (as teacher-engineer). The author designed a professional development plan in cooperation with the teachers. Starting from the fall 2005, every two weeks he conducted didactical engineering seminars (one and a half hour sessions) for teachers. In total, during the academic year, about 20 seminars were held, which included but were not limited to the following diverse activities:

- Analysis and development of learning objectives and expected outcomes for Mathematics topics in the curriculum;
- Detailed analysis of the content and teaching methods;
- Selection and design of tasks, problems, projects and activities for the development of students' mathematical proficiency;
- Design of lesson plans and didactical approaches to the development of students' abilities to reason and solve problems;
- Classroom observations by peers followed by analysis;
- Analysis of student work to identify and address common students' misconceptions;
- Analysis of video records of mathematics classes, etc.

The study continued during four academic years in 2005-2009. The critical point of the study was the first year (2005-2006), when teachers' attitude started slowly changing from passive to neutral. In 2006-2007 academic year, student achievements began to improve and reached the average for the state. The performance measure used in the study was the students' rate in the state standardized test — TAKS (Texas Assessment of Knowledge and Skills). During 2007-2008 academic year the school pupil performance exceeded the average rate for the state of Texas. This year teachers' attitude changed radically. Teachers no longer blamed students and became more optimistic about the results of their work. During the next academic year (2008-2009) the student achievement exceeded the psychologically meaningful level for the school — 85%. The dynamics of the school's student achievements in mathematics compared with the state average for the period of the study is shown in Figure 1.

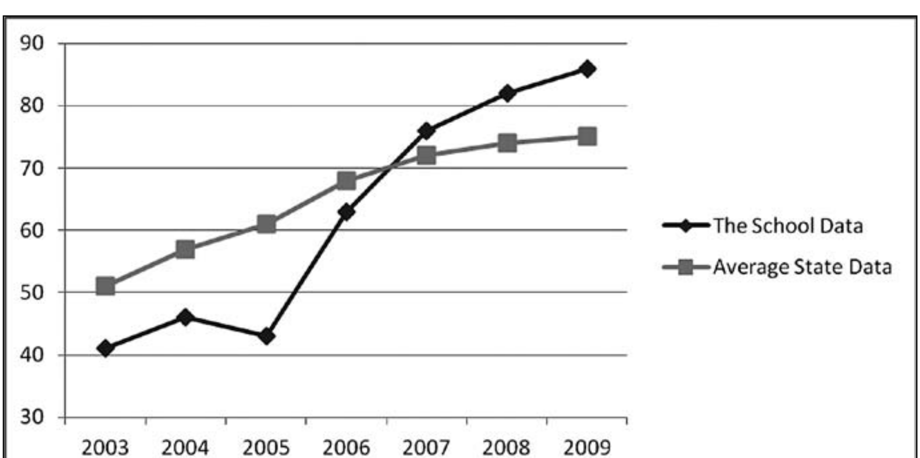

Fig. 1. Dynamics of student achievements in 2005-2009 academic years

The author would like to express special gratitude to the teachers from the school where the study was conducted: Sue Spotts, Ricardo Bombara, Roger Carrera, Michael Garcia, Marcy Loya-Griswell, Elsa Nunez, and others for the productive cooperation.

In this book, the author shares his experience of practical application of didactical engineering of student learning. The book consists of four chapters and an appendix and focuses on the transformation of teaching in the era of new technologies. The digital age demands revision of traditional teaching and learning. Rapidly growing technological innovations in education force a paradigm shift from traditional teaching to engineering of learning. The main emphasis of the book is on understanding and designing the key features of learning experiences (e.g., objectives, content, assessment) through the use of Information and Communication Technologies.

Chapter 1 addresses the transition of the traditional concept of didactics (e.g., didactical triangle: content — teaching — learning), through technologically-enhanced didactics (e.g., didactical tetrahedron: content — teaching — learning — technology) to e-Didactics and didactical engineering as an emerging field of study and design of innovative learning experiences and environments.

The society is experiencing truly revolutionary changes due to the intensive implementation of new digital technologies that provide unprecedented *democratization of knowledge and access to open education*. According to some estimates, more than half a million personal computers and other mobile devices (tablets and cell phones) are now connected to the global network. We are witnessing the formation of a new phenomenon — a virtual learning community — which now includes more than a billion users. The number continues to grow. Along with the trend, the market of distance learning services is steadily increasing, which requires rethinking of traditional teaching. Expansion of these services necessitates training of "online" educators who are capable of analyzing information resources, designing distance courses, and constructing effective learning experiences and environments. Many universities around the globe have established consortia and special platforms to design and offer the so-called MOOCs (massive open online courses) to develop new instrumentation systems to support distance learning, to create databases of multimedia lectures, online courses, e-books, digital libraries, etc. Under these circumstances, the traditional understanding of didactics as a science and an art of teaching does not meet the requirements of the rapidly growing information society. Similar to the paradigm shift from traditional teaching to engineering of learning, we are witnessing the shift from the traditional didactics toward the digital age didactics — e-Didactics — with major emphasis on its engineering function. We call this phenomenon — didactical engineering. The character "*e*" in e-Didactics means more than just "electronic". Expanding the new acronym, we consider "e" in e-Didactics within the framework of *e*ngineering design and 5e model (Bybee et al., 2006), which describes a social constructivist learning cycle, helping students to build new understandings and develop ideas from prior experiences through the following five stages: *e*ngagement, *e*xploration, *e*xplanation, *e*xtension, and *e*valuation. The 5e model could be effectively used for engineering of learning in different formats: face-to-face, blended/hybrid, and distance learning. By synthesizing the transformation, we argue that digital age didactics is simultaneously science, engineering, art of learning and teaching. As an emerging field, didactical engineering offers opportunities for study and design of effective learning experiences and environments in the digital age.

The main goal of Chapter 2 is to introduce Learning Sciences as a new approach to understand learning in the ICT era. “Learning sciences is an interdisciplinary field that studies teaching and learning” (Sawyer, 2006). This emerging innovative field includes but is not limited to multiple disciplines such as cognitive science, educational psychology, anthropology, computer science, to name a few. The Learning Sciences help educators to design effective learning experiences and environments, including distance learning, based on the latest findings about the processes involved in learning. Engineering of effective learning is grounded on the following guiding principles: building on students’ prior knowledge and experiences; developing students’ procedural fluency within the conceptual framework; engaging students in continuous self-regulative, metacognitive, and reflective thinking (Donovan & Bransford, 2005). The Learning Sciences heavily capitalize on theories of constructivism (J. Piget and L. Vygotsky) and constructionism (S. Papert). The Learning Sciences provide guidance to teachers in preparing students to participate in a global society which is increasingly based on technological innovations.

Chapter 3 outlines the toolkit for engineering of learning: the design of learning objectives, the issues related to cognitive tutoring, representations, new literacies, the research-based strategies of learning and teaching, and assessment of learning outcomes. It describes the hierarchy of learning objectives based on the pioneering work of Bloom (1957) as well as its modifications developed by Gerlach and Sullivan (1967), Guilford (1967), de Block (1975), Smith and Stein (2001), et al. The chapter further expands the concept of cognitive demand. In order to engineer effective learning, teachers need to persistently check for students’ understanding to support student learning through the use of a variety of tasks (Shepard et al., 2005). According to Boston and Smith (2009) “different kinds of tasks lead to different types of instruction, which subsequently lead to different opportunities for students’ learning”. This approach focuses on the framework that distinguishes between different levels of cognitive demand: memorization, procedures with and without connections, and reasoning. More specifically, the tasks at the level of memorization involve reproducing previously learned facts, rules, formulae, or definitions. This level may also include committing facts, rules, formulae, or definitions to memory. The tasks at this level “cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure” (Smith & Stein, 1998). Usually such tasks have no connection to the meaning of facts, rules, formulae, or definitions. The procedures without connection are algorithmic by nature and require limited cognitive demand for completion. Moreover, such tasks do not require connections to the concept or meaning that underlie the procedure. The procedures with connections focus students' attention on understanding of concepts and ideas. Such tasks usually are represented in multiple ways (e.g., numerical, visual, concrete, symbolic) and require making connections among multiple representations. The highest level of cognitive demand — reasoning — requires non-routine, non-algorithmic thinking to explore and understand the nature of concepts, processes, or relationships. Such tasks usually require students to access relevant knowledge in order to solve a problem, to examine the constraints that might limit possible solutions. The reasoning tasks demand considerable cognitive effort due to unpredictable nature of the problem solving process at this level.

Chapter 3 also addresses the digital age assessment. Assessment is considered as one of the key didactical components that directly affects the effectiveness of learning. In other words, successful learning largely depends on how well assessment is designed and connected to learning objectives and content. Engineering of assessment could be based on different learning attributes such as: outcome (outcome-based assessment); standards (standards-based assessment); competency (competency-based assessment); performance (performance-based assessment), etc. Regardless of a particular learning attribute, engineering of assessment should consider opportunities for integration of qualitative and quantitative methods of assessment, transformation of extrinsic subjective evaluation into intrinsic objective self-assessment, and development of students' self-monitoring skills for life-long learning.

The ultimate goal of digital age assessment is to strengthen student's responsibility for the process and outcome of self-learning. Digital age requires radical revision of the traditional philosophy of assessment: from discrete assessment — to continuous assessment; from fragmented — to systematic assessment; from single way of assessing student learning — to multiple ways of assessment; from predominantly quantitative — to qualitative and mixed assessment; from fixed — to flexible assessment; from standardized — to authentic assessment; from external — to self-assessment. Examples of portfolio (including e-folios), peer assessment, and other assessment techniques used in e-learning are also discussed in the chapter. Last but not least, the connection between objectives, content, and assessment is considered as an important component of engineering of effective learning.

Chapter 4 focuses on the engineering of digital content. ICT has dramatically changed content representation and delivery. The learning content is no longer a plain text. The digital content is a hypertext with images, videos, 3D objects, and other interactive media types. The chapter describes the modular design approach with its application to the content development: dynamic visualization and animation (including applets), video streaming (e.g., NBC Learn), screen casting (e.g., Khan Academy), gamification (e.g., Quest to Learn). The chapter also addresses the phenomenon of new literacies with emphasis on social practice perspective (Lankshear & Knobel, 2006), which refers to "new socially recognized ways of generating, communicating and negotiating meaningful content through the medium of encoded texts within contexts of participation in discourses".

The chapter discusses the use of digital technologies as the means of engineering (e.g., producing, sharing, and accessing) of interactive content. The content development and modular design are further used in the chapter to discuss the framework for engineering of distance learning.

The photographs and pictures used in the book are uploaded from open resource repositories or belong to public domain. The author would greatly appreciate comments and suggestions sent to mouratt@utep.edu.

Chapter 1

e-Didactics: Digital Age Didactics

This chapter addresses the following main issues:

- what is didactics and where it comes from;
- didactical triangle and didactical tetrahedron;
- what is didactical engineering and why it is important in digital age.

1.1. The Origins of Didactics

People often have limited understanding of didactics; its misinterpretation as teacher-directed learning occurs in some English-speaking countries (Hamilton, 1999, Nordkvelle, 2003). Didactics plays an important role in defining the main construct of this book — engineering of learning — via the following theoretical chain: didactics — didactical engineering — engineering of learning. Thus, let us define didactics through the historical analysis of its origins.

There is a saying “Didactics is as old as times”. It is clear that the need to learn and transmit the experience of previous generations to the next generation is a necessary condition for the development of society. Generally speaking, when one person teaches another person, this situation already suggests didactics. For example, for the case when in the most ancient times senior members of a tribe instructed young fellows in hunting mammoths, using the modern language of didactics the roles can be assigned as follows: the senior — a teacher, the younger members — students, and hunting — the content of teaching and learning. The triangle “teacher — learner — content” is called a didactical triangle. Moreover, the original meaning of the word “didactics” (from the Greek didaskein) is “to teach” or “know how to teach.”

Let us make a brief excursion in the history of didactics in the context of the conceptual origin of didactics. Many authors in the history of education claim that didactics was first proposed by Jan Amos Komensky (Comenius, 1592-1670) — the author of the famous “*Didactica Magna”*. Not diminishing the invaluable contribution of Jan Amos Comenius to the formation of didactics as a science, let us try to restore historical justice.

As noted above, the root of the word “didactics” (“didaskein”, “didascalia”, “didascalica”) is of Greek origin. The term was first used in relation to the choir rehearsals in Ancient Greece (Illich, 1995). The term «didaskaleion» was used for the place where the music teacher conducted these rehearsals (Myhre, 1976). Five hundred (!) years prior to Comenius, in 1120, the French philosopher Hugo of St. Victor published a book called “Didascalicon” (1961), which was recognized as an attempt to improve higher education in the Renaissance Era (Grabmann,1998). In his book, Hugh formulated the framework of educational planning at universities and suggested the rules of systematic teaching and learning using the methods of dialectics (Nordkvelle, 2003).

In ancient Rome and the Hellenistic era in Greece, there was a range of academic disciplines related to Fine Arts. According to the founder of the medieval encyclopedia Isidore of Seville, this set of disciplines included two cycles: the trivium (grammar, dialectic and rhetoric) and the quadrivium (arithmetic, geometry, music, astronomy).[1] We must admit that since the Antiquity there was a kind of confrontation between the two classical fine arts: the dialectic and the rhetoric.

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[1] Isidore of Seville is the patron saint of computers, computer users, and computer technicians.

In ancient Greece, dialectics was the method of philosophical inquiry. This method has gained worldwide recognition through the dialogues of Socrates.

Rhetoric is the art of public speaking. In ancient Greece, and the more so in ancient Rome, preference was given to the rhetoric, although Aristotle has called for "equal rights" for the dialectic and the rhetoric. However, during the Renaissance Era the dialectic "took revenge" over the rhetoric, which is reflected in «Didascalicon» by Hugh. This text was used as a basic manual in the European higher education institutions for the next three or four centuries. The dialectic has reached its dawn in the Middle Ages. Figuratively speaking, if the Antiquity is the golden age of the rhetoric, the Middle Ages is the golden age of the dialectics (Fefner, 1982).

Attention to the problems of education had risen during the Renaissance era, which along with other great achievements was characterized by the rapid development of higher education: the number of universities and, respectively, the number of students in European countries increased considerably. By that time the society accumulated social experience and knowledge that had to be transmitted to the next generations but the lack of transmission mechanisms became an obstacle in the development of the society. In the 12th century, this contradiction, along with the development of higher education in the Renaissance era, to some extent, stimulated the interest in Hugh and his colleagues to study the problems of teaching and learning.

In the 16th century, Pierre de la Ramee (Petrus Ramus), French philosopher and professor of the University of Paris, together with his fellow humanists Rodolphus Agricola and Philip Melanchthon continued the work of Hugh. Their contribution was extremely important for further formation of didactics: the ancient Greek concept of dialectics was gradually transformed into the art of teaching. Melanchthon considered dialectics as a method of teaching properly, orderly and understandably (Ong, 1974). Ramee expressed this idea in a more succinct way: dialectics is an art of teaching. Ramee's vision of the new nature and the role of dialectics in teaching was a kind of predecessor of didactics. In other words, with a certain degree of historical accuracy one can say that didactics emerged from dialectics.

The progressive views of the 16th-century French humanists extended to the whole Europe (Hotson, 1994) undoubtedly had a positive impact on the minds of other European scholars including Wolfgang Ratke. Due to the support of his colleagues Junge and Helwig, in 1612-1613 Ratke made a proposal for an initiative called "didactics as the new art of teaching", which was supported by the Academic Council of the University of Giessen.

Then, in the mid-17th century, Jan Amos Comenius, Czech educator, humanist, and intellectual, presented didactics as a system of knowledge, setting out the basic principles and rules of teaching in his seminal work "The Great Didactic" (1657). The history of didactics after Comenius is well documented in the educational literature.

As to the definition of didactics, it is most often defined as a theory of teaching and learning. Didactics addresses issues related to the main goals and guiding principles of learning and teaching, curriculum, content and methods of teaching and assessment, to name a few. Being asked whether didactics is a science or art of teaching and learning, most readers, based on the traditional definition of didactics, would answer "science." Indeed, as a science, didactics has its categorical apparatus, methods of research, mechanisms to identify trends, its structure and logic. Thus, didactics evolved as a scientific discipline — the theory of teaching and learning.

However, a theory without practice is blind. Didactics needs a teacher who would implement the theory into practice. Here comes "art" part, which plays the vital role in teacher professionalism, teacher personal qualities, culture and teaching style, creativity and talent, teaching philosophy, etc. We also cannot disregard the fact that the founders of the dialectic-didactics Hugh, Ramee, Ratke, and Comenius considered didactics, above all, as an art of teaching. Some scholars understand didactics as the theory, others — as the art of teaching and learning. Both groups are right, in their own way. As a part of pedagogy, didactics is not only the science but also the art of teaching and learning. The evolution of views on didactics is shown in Figure 2.

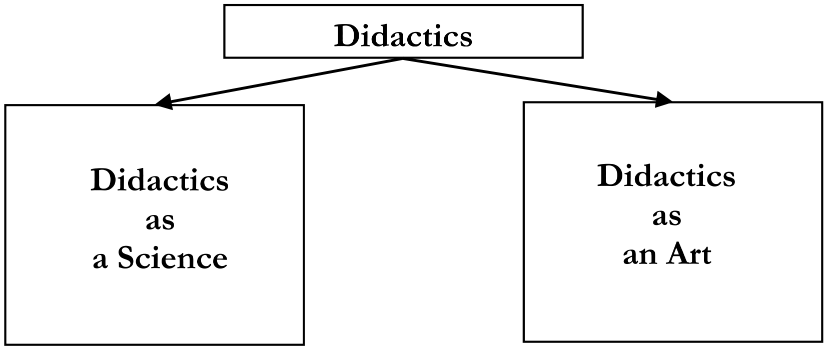

Fig. 2. The evolution of the concept of didactics

The two interpretations of didactics cannot exist separately; therefore, a question about a link between them arises naturally. The diagram does not show a «bridge» between the two components, though potentially this «bridge» should allow teachers to effectively use the didactics-science in the educational practice. To be able to teach effectively, a teacher needs to be able to conduct a comprehensive and meaningful analysis of the teaching processes and situations. S/he must also be able to select, design, and implement a variety of didactical products (e.g., learning objectives, content and learning activities, assessments, etc.).

In addition to being science and art, didactics should also be considered as an engineering activity. Engineering is the process of analysis, design and construction of facilities/mechanisms for practical purposes. Generally, the term «engineering» is applied to buildings and constructions. To build a house, one needs to make calculations for a construction site, economic analysis, including estimated cost of building materials, resources, and labor, etc.), then make the design (the drawing plan) and only then proceed to the construction. In the case of didactic engineering, we are talking about the analysis, design and construction of teaching products for learning. In other words, in addition to science and art didactics should be an engineering of teaching and learning. Therefore, we propose to define *didactics as science, engineering and art of teaching and learning* (Figure 3).

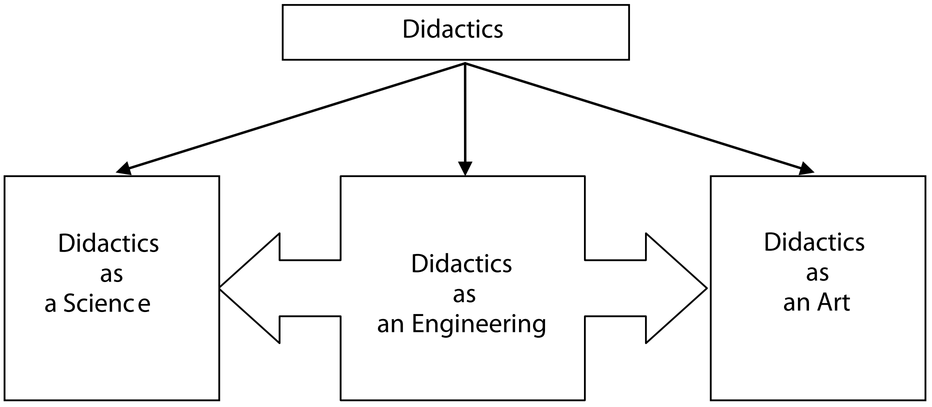

Fig. 3. Didactics as a science, engineering, and art of teaching and learning

1.2. Didactical Triangle and Didactical Tetrahedron

The Traditional View: Didactical Triangle

In a broader sense, a triangle "teacher — learner — content" including interactions among components of the trivium is called didactical (Figure 4). Originally, this construction appears in the work of Chevallard (1982), Brousseau (1997) et al. Chevallard introduces the construction of the didactical system which involves "three components — the teacher, the students, the knowledge taught — and the interactions between them" (Chevallard, 1982: 8). Similar construction to represent the classroom culture system was proposed by Brousseau (1997), which includes the teacher, the student and the *milieu* (e.g., learning tasks, instructional materials, and teaching strategies). Overall, "the didactic triangle in which the student, the teacher, and the content form the vertices (or nodes) of a triangle is the classical *trivium* used to conceptualize teaching and learning…" (Goodchild and Sriraman, 2012: 581).

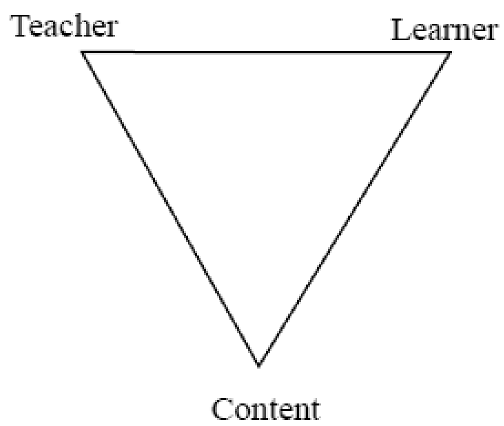

Fig. 4. The traditional didactical triangle

Some scholars are concerned with the limitations of the classical view and suggest to consider the contextual factors (e.g., curriculum, assessment, and classroom culture) in the model. Schoenfeld (2012) claims that "classrooms are cultural systems" (p. 598) and what occurs in mathematics classrooms "is indelibly a function of the cultural forces that shape them — e.g., how curricula are defined and which curricula are made available, how factors such as testing shape teachers' and students' decision making within the classroom" (Ibid.: 598). The revised model of the didactical triangle including the context (as a broader notion incorporating curriculum, assessment, culture, etc.) is presented in Figure 5 below.

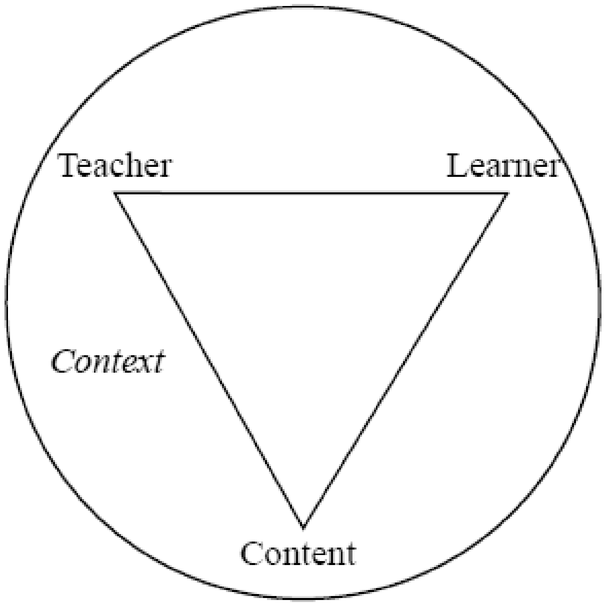

Fig. 5 The didactical triangle within the context

Transformation of the Traditional View: Didactical Tetrahedron

The beginning of the XXI century is characterized by the revolutionary changes associated with the intensive use of new technologies in education. The global web is increasingly impacting the daily lives of individuals and the society. Scholars started recognizing the transforming effect of technology on teaching and learning in the mid-1980-ies, as soon as computer software provided means to represent concepts in multiple ways including graphs, spreadsheets, dynamic visualization tools, etc. (Tall, 1986). Due to the continuous intensive use of new technologies in the learning process, the beginning of the XXI century was marked by the attempts to revise the subject, the goals and objectives of didactics. Thus, "there have been various proposals to expand the heuristic device of the didactical triangle to form a didactical tetrahedron by adding the fourth vertex to acknowledge the significant role of technology in mediating relations between content, student and teacher" (Ruthven, 2012: 627) as shown in Figure 6.

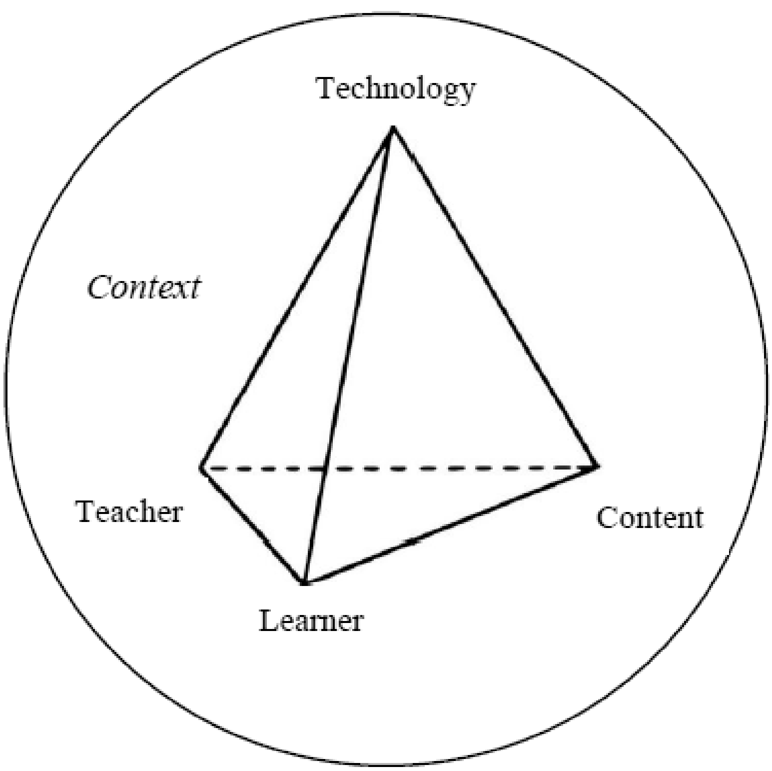

Fig. 6. The didactical tetrahedron

Despite the fact that the didactical tetrahedron represents a whole, each of its faces reflects a specific relationship. For example, the face lying in the bottom of the tetrahedron shown in the figure above represents the traditional didactical triangle "teacher — learner — content". The face «learner — content — technology» reflects the interaction between the student, the content, and the technology that might be called *e-Learning*. This could also imply the 'flipped classroom' approach (Bergmann and Sams, 2012) and the self-organized learning model in a virtual environment (Mitra, 2005). The face "teacher — content — technology" coincides with the face of e-learning; the only difference is that a student is replaced by a teacher. That is why it is called *e-Teaching*. The next face "teacher — learner — technology" reflects an interaction between teachers and students beyond the subject domain via the use of ICT. An example of such interaction could be *e-Advising*. The didactical tetrahedron also depicts the integration of technological, pedagogical, and content knowledge, which is known as TPACK (Koehler & Mishra, 2009).

1.3. e-Didactics and Didactical Engineering

Reconceptualization of the traditional didactics is important in the light of rethinking its role in the digital age towards engineering of learning. New didactics of e-learning is called *e-Didactics* (D'Angelo, 2007).

Broadly defined, e-Didactics is an ICT-integrated didactics. In order to identify its key characteristics, let us summarize the evolutionary development of didactics. As we mentioned earlier (see Chapter 1.1), didactics had several cornerstones in its development. We consider the following main stages in the conceptual evolution of didactics:

- Pre-didactics
- Didactics-dialectics
- Classical didactics
- Digital age didactics.

The pre-didactics stage (IV BC — VII AD) began with Socratic Dialogues written by Plato (IV century BC), which later transformed to the well-known Socratic Method of Teaching. At this stage the classical Fine Arts curriculum was established, which included two major blocks of academic disciplines (e.g., trivium and quadrivium), described later by Isidore of Seville in "Etymologies" (VII AD).

The didactics-dialectics stage (XII — XVI AD) began with the distinguished work of Hugh St. Victor "Didascalicon, or On the Study of Reading" (1120) and further continued with "Dialectique" (1555) by Ramee, where dialectics was considered as an art of teaching.

At the next stage — classical (or traditional) didactics (XVII — mid-XX), we observe an important transition from the art to the science of teaching and learning. The stage of classical didactics began with an initiative proposed by W. Ratke to call "didactics as new art of teaching" (1613) and further developed in "Didactica Magna" (1657) by J.A. Komensky, who outlined the didactical theory as a field of study of teaching and learning. This classical tradition continued to the XX century.

The stage of digital age didactics (late XX — present) began with reconceptualization of classical didactics in the era of Information and Communication Technologies. In 1991, M. Artique proposed didactical engineering as a research and development tool to study teaching and design effective learning. We consider didactical engineering as a turning point from the classical didactics to e-Didactics. In 2007, G.D'Angelo described an e-Didactics paradigm to address the phenomenon of e-Learning.

The conceptual evolution of didactics is presented in Figure 7 below. Comparing the stages of pre-didactics and didactics-dialectics, one can see the emergence of the first teaching method (e.g., Socratic dialog) and curriculum (e.g., classical Fine Arts) as well as recognition of dialectics as an art of teaching. Comparing stages of didactics-dialectics and classical didactics, one can observe the emergence of didactics as a theory and field of teaching and learning study. The key distinction of digital age didactics from the classical didactics emerged in the late XX century through the development of its engineering/design characteristics.

Next, let us compare the key characteristics of classical/traditional and e-Didactics. Didactics still has its main goal on quality of teaching and learning through developing the desired level of students' competency and proficiency. Classical didactics and e-Didactics share similar theoretical foundation based on learning theory and its guiding principles of learning (Bransford, Brown, Cocking , 2000).

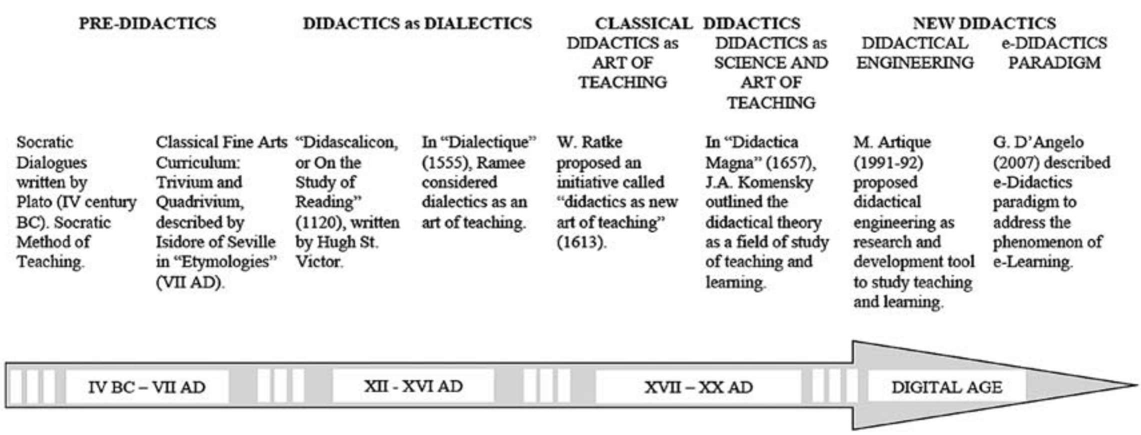

Fig. 7. Conceptual evolution of didactics

The difference between classical/traditional didactics and e-Didactics is stipulated by a paradigm shift in the primary focus of didactics: from teaching to engineering of learning. This shift becomes more visible in online education where teaching in a traditional sense is limited by the structure of the format: if in traditional didactics primary delivery format is face-to-face (and hybrid, in some cases), in e-Didactics — it is mostly blended and online. Along with the changing delivery format, the learning and teaching space is changed: classroom is replaced by the virtual space represented by various learning management systems (LMS) and social networks. Moreover, there is a significant change in the role of a teacher in the digital age didactics: from a transmitter of knowledge to an engineer of student learning. In turn, transformation of teacher's role influences the change in student role: from an information receiver to a connected learner.

Another critical difference could be observed in the primary mode of learning: passive learning transforms to more active and interactive discovery-type student engagement. This difference is impacted by the change in representation of instructional material among others: text and graphics in traditional teaching are replaced by hypertext and media in on-line education; instead of hardcopies, teachers and students get used to deal with softcopies of instructional materials; the principle of visualization in traditional didactics (e.g., mostly static and illustrative) is transformed to the principle of dynamic and interactive visualization in e-Didactics.

The primary mode and means of classroom communication and assessment are also experiencing a significant change: from verbal to written; from oral classroom discourse to written exchange of ideas via online discussion, chats, and social networks; from traditional training and instruction to screencasting and videostreaming; from paper-and-pencil to e-assessment and e-folios, etc.

Last but not least, revolutionary change in information access occurs: from access limited by a textbook and a teacher in traditional didactics to open access to knowledge unlimited by ICT resources in e-Didactics. Table 1 summarizes the results of comparative analysis between traditional and e-Didactics.

Table 1. Comparing traditional didactics and e-Didactics

Characteristics	Traditional Didactics	e-Didactics
Dominating focus	Science and art of teaching	Engineering of learning
Primary goal	Quality of teaching and learning, students' competency and proficiency	
Theoretical basis	Research-based guiding principles of learning	
Delivery format	Face-to-face, hybrid	Hybrid, online, e-Learning
Primary teacher's role	Transmitter of knowledge	Engineer of learning
Primary student's role	Information receiver	Connected leaner
Dominating mode of learning	Passive, active	Interactive
Primary learning and teaching space	Physical classroom, auditorium	Learning management systems, virtual space
Instructional material representation	Text, graphics	Hypertext, media
Instructional material format	Hardcopy	Softcopy
Use of graphics and visualization	Static and illustrative	Dynamic and interactive
Dominating mode of communication	Verbal	Written
Primary means of communication	Classroom discourse	Online discussion boards, chats, social networks
Information access	Limited by the textbook	Unlimited by ICT resources
Primary mode of scaffolding	Training, instructing	Screencasting, videostreaming
Dominating assessment format	Paper-and-pencil assessment	On-line assessment, e-folio

The table shows that e-Didactics has a number of characteristics that makes its position conceptually distinct from the classic didactics. The most important distinction is that ICT and engineering play critical role in e-Didactics. Therefore, *e-Didactics* could be defined as *ICT-integrated didactics with a focus on engineering of learning*.

Within the framework of e-Didactics, we consider the following levels of ICT use: low, medium and high. The low level of ICT is characterized by the spontaneous use of some technological means such as calculators (including graphic calculators) or basic software (e.g., Word, Power Point, Excel). The medium level involves technology-enhanced learning with broad use of ICT and multimedia. The high level includes the use of Learning Management Systems to support the process of e-learning and e-teaching. The format of teaching and learning is divided into traditional (f2f = face-to-face), hybrid (or blended) and distance (online). The level of interactivity includes passive, active, and interactive learning environments. The interactive level is characterized by engaging students in constructive learning experiences (Bybee et al., 2006). If the zone of traditional didactics is the low level of technological tools usage in a predominantly f2f teaching and learning with primarily passive and some active learning, the zone of e-Didactics goes beyond the traditional boundaries of teaching and learning toward the virtual space using digital tools, interactive multimedia and systems of distance education with predominately interactive learning environments (Figure 8).

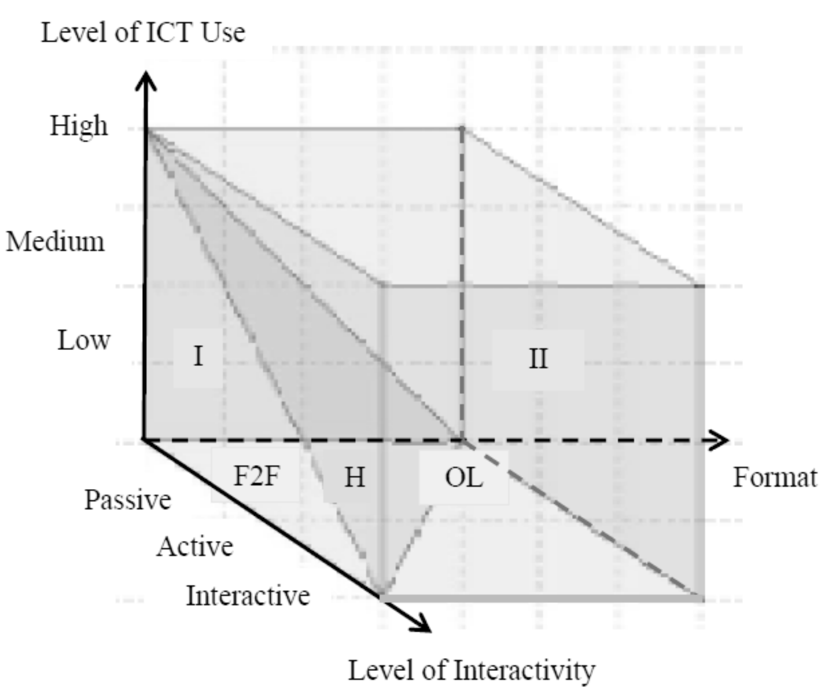

Fig. 8. The zones of traditional and e-Didactics:
F2F — face-to-face; H — hybrid; OL — online;
I — zone of traditional didactics; II — zone of e-Didactics

Engineering plays a significant role in the design of learning objectives, development of content, selection of teaching and learning means, design of assessment in the structure of the e-Didactics.

What is Didactical Engineering?

There are genetic engineering, computer engineering, or social engineering, to name a few. How these different 'engineerings' are defined? For example, genetic engineering is defined as a set of molecular biology and genetics techniques associated with the analysis, modeling, and design of new combinations of genes. Computer engineering is related to the analysis, software development and integration of software with a variety of computer platforms, hardware, and systems. Social engineering is defined as a design of activities for new social institutions as well as restructuring of existing social institutions by gradual reform and change.

Each of the above cases involves, to a certain extent, the following elements of engineering: analysis, design, modeling, construction, and development. In a broader sense, engineering is defined as analysis, design and/or construction of facilities for practical purposes. Consequently, engineering as a human activity may be applicable to various professions and it involves a wide range of activities from the analysis and design of facilities to their operation and maintenance. Therefore, didactical engineering is a kind of a generalized concept of the engineering approach to didactics.

Didactical engineering is a relatively new approach in modern education. That is why there are few publications in this area. First attempts to use an engineering approach in didactics took place in the 1990-ies (Artigue & Perrin-Glorian, 1991; Artigue, 1992; Douady, 1997). Douady (1997) defines didactical engineering as a series of teacher-engineer related didactical actions, which ensure the implementation of the learning project with a group of students. Ruthven (2002) believes that "didactical engineering aims to develop highly precise designs that will be reproducible under suitably controlled classroom conditions, and to do so through systematic and exhaustive analysis of variables and strategies, framed in terms of an overarching didactical theory" (p. 586).

Didactical engineering aims at using research-based practices and promotes the development of teaching design thinking. Didactical engineering also fosters the development of teachers' analytic reasoning focused at the implementation of macro and micro analysis of didactical systems, processes and situations. Accordingly, didactical engineering (Figure 9) has its own subject domain that is characterized by the following main parameters:

1. analysis, design and construction of outcome-oriented teaching products (e.g., learning technologies);
2. application of a scientific method and design thinking to the analysis of didactical systems, processes and situations in order to create effective learning environments.

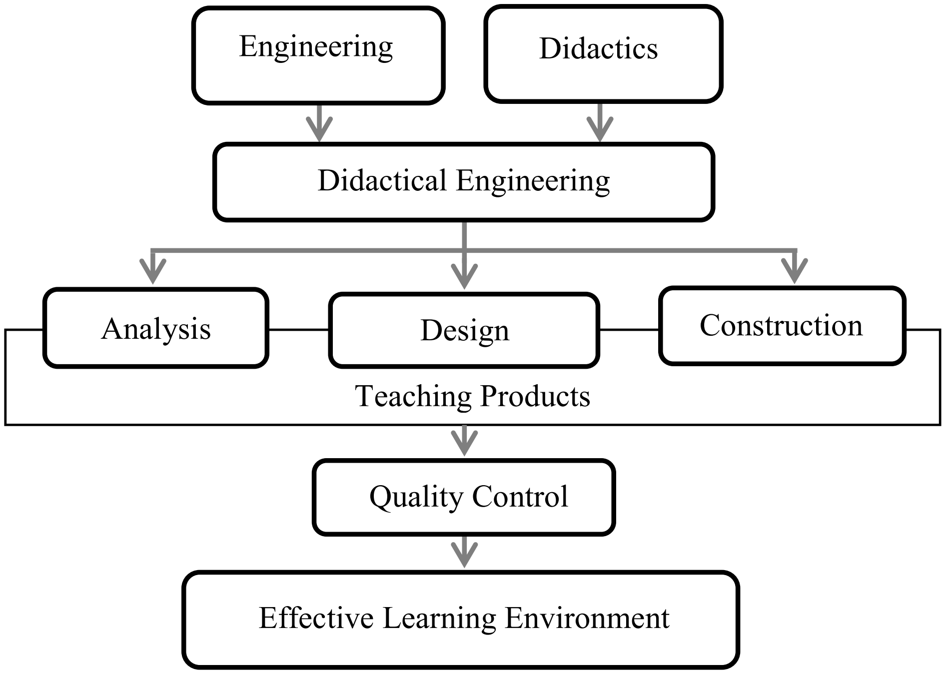

Fig. 9. Didactical engineering and its main purpose:
the design of effective learning environment

Didactical engineering has a dual nature: it is both a product and a process of an educational design activity. It is a product of didactical analysis and design as well as the process of applying an engineered teaching product to the learning environment. Thus, as an instructional activity, didactical engineering can be defined as a series of steps in analyzing, designing, and constructing of teaching products and their use in the instructional process in order to create effective learning environments and achieve desired learning outcomes.

A Teacher in the Digital Age: Teacher-Engineer

The market of online educational services has been steadily growing. For example, in the Department of Teacher Education at the University of Texas at El Paso (USA) about 50% of graduate courses are delivered in an online format.

With the purpose of expanding online services, the leading universities create the MOOC consortia (e.g., Coursera, Udacity, edX) to initiate special programs for supporting the design and delivery of online courses as well as the development of new tools for online learning systems (Grainger, 2013; Yuan & Powell, 2013). However, some skeptics claim that massive open online courses are not a panacea. The Gallup/Inside Higher Ed conducted a survey of the presidents of several US universities involved in offering MOOC. The major findings of this survey is that 54% of the participants somewhat disagree or are not sure whether MOOC foster creative pedagogical strategies. Moreover, 83% of the participants disagree or are not sure that MOOC improve the learning of all students (Jaschik, 2013).

Some colleges such as the Duke University and the Amherst College rejected proposals to join the MOOC consortia because the faculty does not see the benefits of MOOC in improving student learning, in particular at the undergraduate level. One of the faculty members expressed her concern about poorly designed online classes saying "students will watch recorded lectures and participate in sections via Webcam enjoying neither the advantages of self-paced learning nor the responsiveness of a professor who teaches to the passions and curiosities of students" (Kolowich, 2013).

The question is "how to make sense of this skepticism with the seemingly endless flurry of the MOOC-related announcements these days?" (Jaschik, 2013). A possible answer may be that the speed at which colleges have embraced MOOCs has little to do with the readiness of the "MOOC industry" to offer high quality products. To do so a paradigm shift should occur: the shift from teaching to engineering of learning, which will foster creative pedagogical strategies to design and implement online courses. And, consequently, this shift develops an urgent need for training of "online" educators who are able to design and deliver effective distance education.

This also creates a domino effect: along with the transfer of many university disciplines, including teacher education courses, to the online format, there is a need to revisit the training of school teachers. Instead of traditional teacher training the focus is shifting toward the new type of teachers who can work effectively in the digital environment and satisfy high demands on teachers' knowledge and ability to engineer online student learning. Moreover, in the digital era a teacher is not just an online tutor. The teacher becomes a kind of analyst and manager of informational resources, designer of courses, modules, lesson fragments using interactive multimedia tools.

The emerging changes in the role of a teacher raise an important question: what kind of teacher is needed in the digital age? According to The National Educational Technology Standards (NETS) developed by the International Society for Technology in Education (ISTE), the advancement of digital age teaching is associated with the following standards: "(1) facilitate and inspire student learning and creativity; (2) design and develop digital age learning experiences and assessments; (3) model digital age work and learning; (4) promote and model digital citizenship and responsibility; and (5) engage in professional growth and leadership" (ISTE, 2008).

The new set of standards was published by UNESCO (2011). The UNESCO ICT Competency Framework for Teachers emphasizes "that it is not enough for teachers to have ICT competencies… teachers need to be able to help students become collaborative, problem-solving, creative learners through using ICT" (UNESCO, 2011: 3). This statement, in a way, echoes the above Duke University faculty's concern on a lack of MOOC's support of students' curiosity and creativity in online learning. The UNESCO Framework addresses the following teacher competencies in the digital age: '(1) understanding ICT in education; (2) curriculum and assessment; (3) pedagogy; (4) ICT; (5) organization and administration; (6) teacher professional learning" (UNESCO, 2011: 3). The UNESCO Framework further expands the significance of integrating ICT and Pedagogy through the following teacher competences: (a) "integrate ICT into didactic knowledge acquisition and learning theory models; (b) create learning activities that use ICT resources to support specific educational outcomes; (c) apply ICT to "just in time" and "spontaneous" learning interactions; (d) design presentations that appropriately incorporate ICT resources" (UNESCO, 2011: 50-52).

Some of the above standards and competences expand the role of a teacher much beyond the traditional teaching. For instance, the UNESCO's competency on integration of ICT and Pedagogy — "integrate ICT into didactic knowledge acquisition and learning theory models" — has a hidden call for the expansion of the role of a teacher to become a teacher-didactician — someone who is knowledgeable of the learning theory and research-based teaching. For a similar reason, Jaworsky (2012) proposed an addition of an extra vertex to the traditional didactical triangle to include the didactician as an integral part of the system for teacher development.

At the same time, the ISTE's standard for teachers on "designing and developing digital age learning experiences and assessments" requires a teacher to extend his/her role as an engineer — someone who knows and able to design and construct effective learning environments. In traditional education, the three roles mentioned above (a teacher, a didactician, an engineer) were isolated.

Obviously, there is an emerging need to train a new teacher to face the challenges of the digital age, to be a teacher, who, to some extent, combines the competences of a didactician and an engineer. The digital age standards and competences demand "a push" for e-Teaching as an integration of the roles of a teacher, a didactician, and an engineer as presented in Figure 10.

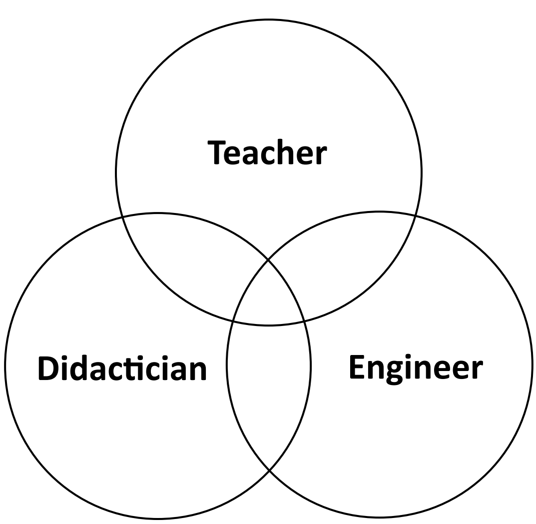

Fig. 10. Teacher-engineer

The integration implies reconceptualization of the key role of *a teacher-engineer* in the digital age: traditional teaching transforms toward *the research-based engineering of student learning*. This transformation requires a teacher-engineer to understand teaching theory and learning sciences (Bransford et al., 2000; Sawyer, 2006) in order to effectively design learning objectives, digital content and assessment, and as to make connections between the objectives, content, and assessment (Figure 11).

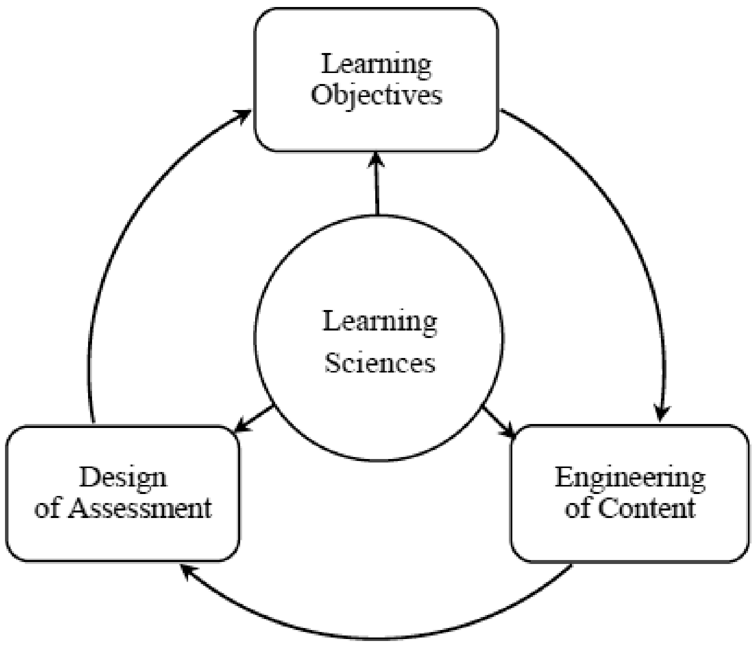

Fig. 11. Learning sciences and engineering of learning objectives, content, and assessment

The engineering of learning paradigm places a critical emphasis on the development of teachers' design thinking (Dym et al., 2005)). The development of teacher-engineer design thinking is a complex process based on the advancements of learning sciences. The teacher-engineer should acquire the following key competences:

1) design of learning objectives: create outcome-based, technology-enhanced learning environments that enable students to set their own learning objectives, monitor and assess their own learning progress;
2) engineering of content: develop interactive content and relevant learning experiences through selection and design of tasks, problems, projects, and activities that incorporates digital tools and ICT resources to promote student learning and creativity;
3) design of assessment: select and develop authentic assessments aligned with learning objectives and content; use assessment data to improve teaching and promote student learning.

Chapter 2

Learning Sciences in the Digital Age

This chapter addresses the following main issues:

- learning patterns and guiding principles of learning;
- constructivism and constructionism;
- social constructivism and learning culture in action.

2.1. The Guiding Principles of Learning

During the last two decades the learning sciences scholars significantly advanced the research in learning theories. Within half-a-decade, the U.S. National Research Council published two major studies "How People Learn" (2000) and "How Students Learn" (2005) with a focus "on three fundamental and well-established principles of learning":

1) Building on students' prior knowledge;
2) Connecting students' factual knowledge and conceptual understanding;
3) Involving students in meta-cognitive and self-monitoring activities.

Let us briefly discuss each of the above principles.

Principle 1. In addressing students' prior knowledge, we are trying to gauge and record students' understanding of previously learned facts, concepts and procedures that would help them to learn new material. "Students come to the classroom with preconceptions about how the world works. If their initial understanding is not engaged, they may fail to grasp the new concepts and information that are taught, or they may learn them for the purposes of a test but revert to their preconceptions outside the classroom" (Donovan & Bransford, 2005: 1). Based on the first principle, it is of major importance to continually make links between students' experiences outside the classroom (e.g., everyday informal experiential out-of-school knowledge) and inside the school and classroom (e.g., school-based or "instructional knowledge").

Building on prior knowledge requires considering certain sequences, for instance, while introducing a new topic it is helpful to start with an activity to assess students' preconceptions and keep building on students' prior understanding and experiences. How can we best do this? One way is to use a powerful instructional strategy — the "bridging context". The bridging context is a context that serves to connect student's experiences through multiple representations, for example, numeric (equations) and spatial (graphical) understandings and to link their everyday experiences to lessons taught. Another possibility is to engage students' everyday experiential knowledge. The experiential knowledge is a knowledge that students learn through their practical experience. Using the language strategically and as a link to more formal language use is another way to build on students' prior knowledge. This does not mean that all problems, tasks, statements should be phrased in the "student language". It is important for students to learn formal terminology and abstract symbolism. However, using the student language is a way of assessing students' knowledge on particular topic and then build on what they already know to guide them toward deeper understanding and use of formal language.

Summarizing, the first guiding principle suggests that students' prior knowledge is a building block for the development of more sophisticated ways of thinking. Topics and activities presumed to be challenging and difficult for students may in fact have intuitive or experiential underpinnings, and it is important to discover these and use them for formalizing student's thinking (Donovan & Bransford, 2005).

Principle 2. This principle suggests the importance of both conceptual understanding and procedural fluency, as well as an effective organization of knowledge that facilitates strategy development and adaptive reasoning (Donovan & Bransford, 2005). In order to implement this principle in a mathematics classroom, a teacher needs to recognize and address the following main strategies.

Developing Students' Knowledge Networks. This strategy requires close link between procedural knowledge and conceptual understanding. The network of knowledge must include both new concepts and procedures. Teaching in a way that supports both conceptual understanding and procedural fluency requires the primary concepts underlying a subject domain to be clear to the teacher or become clear during the process of teaching for proficiency. Due to the fact that some subjects, including mathematics, for instance, have traditionally been taught with an emphasis on the procedure, the teachers who were taught this way might initially have difficulty identifying or using the core conceptual understandings in a subject domain. Therefore, teacher training with the focus on these guiding principles is the key component of effective implementation of the principles in the classroom.

Addressing Students' Learning Paths. The above networks of knowledge could be often organized as *learning paths* from informal concrete methods to abbreviated, more general, and abstract methods. The knowledge of student learning paths and knowledge networks helps teachers to direct student learning along productive lines toward valued knowledge networks. The research on learning has uncovered important information on a number of typical learning paths and knowledge networks involved in acquiring knowledge about a variety of concepts (Donovan & Bransford, 2005). As teachers guide students through learning paths, a balance must be maintained between learner-centered and knowledge-centered needs. The learning path of the class must also continually relate to individual learner knowledge.

Using Multiple Methods. The discussion of multiple methods in the classroom — drawing attention to why different methods work and to the relative efficiency and reliability of each method — can help to provide a conceptual ladder that helps the students to move in a connected way from where they are to a more efficient and abstract approach (Donovan & Bransford, 2005). This view of mathematics which involves different methods does not mean that a teacher or a curriculum must teach multiple methods for every domain. Alternative methods might frequently arise in the classroom, either because students bring them from home or because students think differently about many mathematical problems. Frequently, there are viable alternative methods for solving a problem, and discussing the advantages and disadvantages of each method can facilitate flexibility and deeper understanding.

Principle 3. This principle is concerned with a way of making student's thinking visible in a learning process. Metacognition is considered to be one of the key approaches to promote student's thinking about their learning. "Learning about oneself as a learner, thinker, and problem solver is an important aspect of metacognition" (Donovan & Bransford, 2005: 236).

The metacognition principle suggests the following instructional strategies to support students' self-monitoring activities: involving students in debugging errors, engaging students in external and internal dialogue, and encouraging students to seek and offer help in challenging learning situations.

Debugging Errors. The National Research Council strongly recommends to facilitate students' metacognitive activities by "shifting from a focus on answers as just right or wrong to a more detailed focus on "debugging" a wrong answer, that is, finding where the error is, why it is an error, and correcting it" (Donovan & Bransford, 2005: 239). Traditionally, debugging errors was primarily the teacher's activity: the teacher would grade student's work, find errors, and report them to students along with the grade. Debugging errors should be shifted to students. Students should develop critical skills to recognize an error, identify it, locate the source of an error, fix it and check the solution for correctness.

Internal and External Dialogue. According to the National Council of Teachers of Mathematics Standards (NCTM, 2005), communication is an important process that enriches student learning. The classroom culture should be built around meaningful content-focused communication and discourse whether it is a reflection on student's own learning and thinking (internal dialogue) or discussion with peers on comparing and contrasting different methods of problem solving (external dialog). "Of course, teachers must help students to interact fruitfully" (Donovan & Bransford, 2005: 241) through modeling good questioning techniques, providing support structure for student learning, creating an atmosphere of subject-specific communication and collaboration.

Seeking and Offering Help. Teacher's acceptance of challenge translates to student productive attitude toward problem solving (Valverde & Tchoshanov, 2013). Therefore, it is critically important to encourage teachers to help students to be independent problem solvers and actively seek for information or assistance when they face a challenging problem. "Students must have enough confidence not only to engage with problems and try to solve them, but also to seek help when they are stuck" (Donovan & Bransford, 2005: 241). At the same time, working in groups in solving challenging problems might facilitate the environment where students can collaboratively offer help to each other in "tough" situations. "Such helping can also increase the metacognitive awareness of the helper as he or she takes into consideration the thinking of the student being helped" (Donovan & Bransford, 2005: 242).

Along with the guiding principles of learning, it is important to consider advances of brain-compatible research in education to support student learning. In the last couple of decades, studies of neuropsychological basis of the learning processes are steadily growing as evidenced by the variety of subject domains involved and a number of papers published during this period (Bruer, 1993; Caine and Caine, 1994; Chabris and Kosslyn, 1998; Dehaene, 1996). That is why the decade of the 1990-ies was called the "decade of brain". One of the interesting challenges is the problem of adapting the advances in neuropsychology and brain research to teaching.

The traditional popular image of the distinction between the functions of the left and right hemispheres is still strong among practitioners and some scholars, particularly, the fact that the left hemisphere is a domain of languages, numbers, logic, analysis, and the right hemisphere is a domain of images, shapes, intuition, synthesis, etc. However, in the light of modern advances in brain research this view occur to be limited and incomplete (Posner and Raichle, 1994). Still there are scholars who use these outdated ideas to propose teacher training on brain-based education (Jensen, 1988; Sousa, 1995). For instance, for the development of the left hemisphere functions Sousa (1995) suggests using different methods of reading, writing, and arithmetic. The development of students' imaginative right hemisphere, according to the same author, requires intensive use of visualization strategies.

In part, this distribution of functions between the hemispheres is based on the structure of the human brain. This simplification cannot be abused and overgeneralized. In fact, the brain functions as a whole and performs certain tasks (verbal or visual) in conjunction with the neural structures located in both left and right hemispheres of the brain. Posner and Raichle (1994) used the following example to study the human brain in the process of solving basic visual spatial problems: "Specify the location of the given two points based on the questions below:

- which point is located higher than the other one?
- is the distance between the points greater than one meter?"

According to the traditional theory, it is a typical "right-brain" problem. However, the experiment showed that the first part of the task is dealing with the categorical spatial reasoning and mostly carried out by active zones of the left hemisphere of the brain, and the second part of the task, directed by interposition of objects, stimulated the neuron populations of the right hemisphere. Moreover, the study showed that the left hemisphere of the human brain may do as good job as the right one in "solving" visual spatial tasks. Based on the results of the study, Posner and Raichle (1994) also claim that the traditional opinion on creative thinking as a function of the right hemisphere of the human brain is inaccurate.

Another revealing example: in accordance with the traditional theory, the elementary school task "What is greater 2 or 5?" is a left-hemispheric arithmetic task. However, research conducted by Dehaene (1996) suggests that in dealing with such problems the human brain functions as a bilingual learner: it "speaks" descriptive language when we say the names of the numbers "two" and "five" and it "speaks" numeric language when we use the symbolic representation of "2" and "5". In the first case, the areas of the left hemisphere are activated, and in the second case — neuronal populations of both hemispheres of the human brain are engaged.

These examples show that depending on the specific conditions of the task, whether it is verbal or numeric, arithmetic or visual, different areas of the hemispheres could be involved in solving the problem. The distinct separation of functions of the right and left hemispheres of the brain is one of the examples of the "myths" that was debunked by the advanced research in the field during "the decade of brain".

The next myth is the scientific cooperation in the study of the brain among neuroscientists and psychologists. For a certain period of time, two seemingly related branches of the scientific knowledge — neuropsychology (the science of brain) and psychology (in this particular case, cognitive science) — have evolved quite separately. Neuropsychology, to put it in computer terms, explored the "hardware" (structure and function) of the brain, whereas psychology independently studied the "software" of the brain (mental mechanisms of cognitive activity). Meanwhile, the educational scholars attempted to use fragmentarily the results of each of the disciplines as a scientific basis for interpretation of the learning process. Only by the end of the XX century researchers managed to merge the advances of these disciplines in integrative brain-compatible education (Bruer, 1993). The emergence of the combined field made it obvious that most of the previous attempts were nothing but the application of a simplified version of neuropsychology achievements to understanding of learning and teaching. At the same time, this approach brought forward an opportunity to formulate a set of principles about brain functioning during the learning process. This set includes the following principles (Springer and Deutsch, 1993; Sylwester, 1995).

Brain is a parallel processor. The human brain is able to perform multiple functions simultaneously. Thinking, emotions, imagination, and other complex processes may occur in the brain at the same time, along with the mechanisms of information processing and socio-cultural interaction (communication) with other people. Based on this principle, the teacher could provide opportunities for the involvement of students in a variety of content and learning activities using different teaching methods and techniques.

Learning is a natural mechanism for the development of brain. Learning is as natural for a human body, in general, and for a human brain, in particular, as respiration. Nature has endowed a human brain as capable of learning and, therefore, curiosity and desire for knowledge are key intellectual needs for brain development. Didactics as a science, engineering, and art of teaching and learning should provide conditions and environment to meet the critical intellectual needs.

Building on prior experience and the search for meaning are innate qualities of the human brain. Brain is always functioning in the communication mode between the previous experience and a new situation. Understanding and comprehension of the new situation occurs when the brain finds a support in the prior knowledge and ideas. Hence, it is critically important to engage students' prior experiences in order to acquire new knowledge (see Principle 1 above). This principle also supports the Vygotskian conception of the zone of proximal development (ZPD) — the distance between what a learner knows and what s/he could potentially learn with the help of "a more knowledgeable one" (Vygotsky, 1978).

Brain looks for a pattern. Confusion and chaos complicate the productive functioning of a human brain. In any given situation, no matter how random it is, the brain "tries" to find patterns. The following task illustrates this principle — "You have a minute to memorize the given number 1123581321345589. After a minute write it down on a piece of paper." At the first glance, the task is meaningless because it seems to have no pattern. However, there is a hidden pattern. In mathematics, this numerical pattern is called the Fibonacci sequence where each successive number is the sum of the two preceding numbers. According to this principle, learning aimed at mere memorization is not productive for the brain development. At the same time, learning aimed at finding patterns is a good "food" for brain. In other words, learning is effective when a student's brain is developed by overcoming intellectual difficulties in searching for a pattern.

Emotion is a necessary factor in the brain development. Surprise, indignation, inspiration, a sense of beauty, and even a sense of humor, to name a few, are permanent "companions" in the process of productive functioning of a human brain. Neuropsychologists claim that emotion and cognition are inseparable. This principle emphasizes an obvious need for inclusion of the emotional background in the learning process via contradictions, paradoxical situations, elements of literature, poetry, music, humor, etc. regardless of the subject specific content, whether it is mathematics, history, language or any other discipline. Subjects learned in a supportive emotional atmosphere are better remembered and understood, as they have more stable relations with the corresponding emotional state. Moreover, the emotional factor stimulates thinking and creativity of the student.

Brain is capable to simultaneously analyze and synthesize an incoming information. The results of neuropsychological studies show that brain has a unique ability to "see" an object as a whole and "recognize" its parts. Brain can learn to divide and multiply at the same time. In other words, the execution of mutually inverse operations is another natural ability of the human brain. Analysis and synthesis are two important and constantly interacting cognitive processes in learning.

Learning aimed at developing students' analytical skills only, or as it is otherwise called — "learning by steps" blocks the natural potential of learner's brain, its innate ability to simultaneously analyze and synthesize the information. The same is true about the so-called "holistic learning", which underestimates the development of students' analytical abilities. With this principle in mind, the learning materials should be presented in a constant interaction between the whole and a part, analysis and synthesis, induction and deduction, direct and inverse methods of solving problems.

Brain is able to operate simultaneously with a focused attention and peripheral perception. A human brain can absorb the information that lies not only in the immediate field of attention, but also beyond it. Thus, a child in the classroom perceives both teacher's words and sounds outside the classroom in the hallway of the school. In a well-organized classroom, a teacher can use the features of a child's peripheral perception as a constructive factor of learning. For example, producers use the background music to enhance the context of the movie. At the same time, if this principle is ignored, the mechanism of peripheral perception could act as a destructive element in the learning process.

Conscious and subconscious processes in learner's brain occur simultaneously. In a learning process we receive a lot more information than we can imagine. It could be compared to an iceberg where the underwater part can be associated with the processes that occur in learning at a subconscious level. Peripheral signals (sounds, words, images, etc.) are often fed into brain "without permission" of our consciousness and submerge into the deepest layers of the subconscious. Reaching the subconscious, these signals can rise to the level of consciousness with a certain delay or indirectly act on the human mind from inside through the inner motives, unconscious desires, feelings and states. In the learning process, this principle should be taken into account in conjunction with other neurophysiological principles. A student is impacted not only and not so much by what a teacher said but also by the full range of internal (prior experience, emotional state, level of motivation, individual characteristics, etc.) and external (atmosphere in the classroom, sound, light, etc.) factors of the learning environment.

Brain memorizes information at different levels: at the level of visual-spatial memory and rote memorization level. The first level is a more natural way of memorization. The second one produces high cognitive load. For example, we have no or little difficulty in restoring a picture of where and how we spent the previous evening. It does not require special ways of storing information, because it is located and coded in our visual-spatial memory system. This system is closely linked with the natural ability of human brain to sensibly perceive and encode the information. The second level is called a rote memorization and it provides us with invaluable assistance in cases when we need to remember isolated pieces of information such as certain dates, names, phone numbers, phrases, etc. The more information is disconnected from our previous knowledge, the greater the cognitive load is. The disadvantage of this system is obvious: knowledge based on rote memorization is not stable and unproductive. In contrast, visual-spatial memory systematizes the information in a brain as in the library and keeps it organized and connected. In this case, one can easily store the information and quickly retrieve it. This implies the following sub-principle: brain understands and remembers best when information is "imprinted" into the visual-spatial memory (the principle of visualization).

Brain functioning is stimulated by freedom and creativity and suppressed by the atmosphere of coercion and threat. It is known that creative persons cannot tolerate any violence on themselves or on others. Neuropsychologists believe that to become a creative person one should be led by another creative person, or a person who is able to create a learning environment that provides freedom for creativity. Some teachers in an effort to maintain strict discipline in the classroom could unconsciously suppress the atmosphere of creativity. Of course, this does not mean that the classroom management contradicts the development of students' creativity. Rather, a creative learning environment naturally eliminates an issue of discipline in the classroom.

The brain of every human is unique. The brain of each human being has its own individual characteristics in terms of information processing, predominance of certain system of memorization, flexibility of mental processes, etc. That is why every human being has his/her own individual style of learning, own unique understanding of the world, own original style of thinking. The task of a teacher is to maintain the uniqueness of each student via recognizing and supporting student's way of seeing, reasoning, and learning. This principle is particularly evident in the philosophy of constructivism (to be further discussed in Chapter 2.2).

Application of the principles of the brain-compatible education in teaching and learning is presented in Table 2.

Table 2. Application of the brain-compatible principles in teaching and learning

Principles of Brain-Compatible Education	Application in Teaching and Learning
Brain is a parallel processor	Variability of teaching and learning methods and forms Learning in small groups and team learning Multiple representations in learning and teaching
Learning is a natural mechanism of brain development	Learning at an optimal level of complexity Use of discovery learning Constructive learning experience
Building on prior experience and the search for meaning are innate qualities of human brain	Use of practical applications and real-life examples Interdisciplinary connections Problem-based learning
Brain looks for a pattern	Patterns and algebraic reasoning Proofs and refutations Use of counter-examples and contradictions in learning
Emotion is the necessary factor in the brain development	Games in learning Use of aesthetic elements in teaching and learning Paradoxes, surprise situations, riddles in learning
Brain is capable to simultaneously analyze and synthesize an incoming information	Use of inverse operation in learning Inductive and deductive reasoning in problem solving Systemic thinking
Brain is able to operate simultaneously with a focused attention and peripheral perception	Creating productive classroom atmosphere Ergonomics of classroom Use of background music
Processes of conscious and subconscious in learner's brain occur simultaneously	Build on previous knowledge and experience Individualized learning Development of students' self-monitoring
Brain memorizes information at different levels: at the level of visual — spatial memory and rote memorization level	Use of visualization in teaching Verbal, symbolic, numerical, visual and other forms of representation Use of cognitive maps
Brain functioning is stimulated by freedom and creativity and suppressed by the atmosphere of coercion and threat	Creative projects Cooperative learning Use of creative thinking techniques (e.g., brainstorming)
Brain of every human is unique	Individualized learning Constructivism in learning Learner-centered pedagogy

Neuropsychologists argue that education which is not supported by brain-based principles is "blind" teaching and learning. It could lead to weakening of the natural mechanisms of cognitive development. In this case, the recovery of these mechanisms or re-teaching will take longer than the process of "natural" learning consistent with the brain-compatible principles. The "decade of brain" is gradually transitioning into the "decade of mind", which provides educators with an ample opportunity to design learning experiences and environments in accordance with the scientific mechanisms of brain's functioning.

2.2. Constructivism

In this section, we provide an overview of the key ideas of constructivism, its basic principles in the context of a learning process, and briefly discuss its advantages and disadvantages.

The key idea of constructivism is that knowledge cannot be simply transmitted to a student. One can only create pedagogical conditions for successful construction of knowledge and understanding. From a philosophical standpoint, constructivism reflects fairly simple fact: each of us constructs his/her own understanding of the world. Thus, each of us has a unique vision of the world, belief, and viewpoint.

Constructivism is a pedagogical theory that gives priority to a learner's point of view no matter how idiosyncratic it might be. According to Jean-Jacques Piaget, student's opinion is a starting position for construction of new knowledge by overcoming the cognitive conflict between the existing internal structure (schema) and external unknown reality. Eliminating this conflict restores the so-called *cognitive equilibrium* (balance) characterized by the processes of *assimilation* of new knowledge into the existing schema and *accommodation* (e.g., change, modification, replacement) of previous schema based on newly learned knowledge and understanding. Another prominent scholar — Lev Vygotsky — added an important social dimension to constructivism emphasizing *co-construction* of knowledge and understanding. Vygotsky's claim that a learner develops new knowledge and understanding through interaction with others expands the theory toward *social constructivism*.

Constructivism values the process more than the result. Piaget argues that scientific knowledge is not a static phenomenon; it is a process, more specifically, the process of continuous construction and reorganization.

Implementation of constructivism in the classroom requires rethinking of traditional instructional practices. For instance, learning objectives and learning outcomes should be designed around the key position of constructivism: knowledge cannot be transmitted to a student; it could be self-constructed by a student or co-constructed in the process of student's interaction with others. That is why constructivists try to avoid the "imposing" terminology in the design of the learning objectives and outcomes, for example, teacher-directed actions such as "teach", "cover", "tell", "show", etc. Instead, constructivism encourages using student-centered language in the design of learning objectives and outcomes: "construct", "engage", "understand", "justify", "reason", "reflect", etc.

Student motivation should be driven by real life exploratory activities, which include but are not limited to searching, investigating, and solving sound socially relevant problems, especially those arising at school, in the neighborhood, within a community (e.g., environmental, economic, social, etc.). These types of problems and activities engage students in data collection, analysis, and problem solving that contribute to the well-being of their immediate environment.

Congruently, the content should be developed around those concepts and ideas that support students' understanding, stimulate students' reasoning, encourage students to share their assumptions, hypotheses and conjectures, motivate speaking out, involve students into meaningful dialogue and exchange of diverse viewpoints. Therefore, the classroom culture and environment should be built with an emphasis on student learning, student intellectual needs, student collaboration, and student success.

This type of student-centered environment is supported by the work of scholars. For example, the framework of the 5e model (Bybee et al., 2006) describes a social constructivist learning cycle, which helps students to build new understandings and draw ideas from prior experiences through the following five stages: engage, explore, explain, extend, and evaluate. The 5e model could be effectively used for engineering of learning in different formats: face-to-face, blended/hybrid, and distance learning.

The main objective of the "*engage* stage" is to engineer student learning via building their intrinsic motivation and involving students in the activity along with conducting pre-assessment of their prior knowledge and understanding. During this stage, students make connections between past, present and new learning experiences. At the "*explore* stage" students are directly involved in an inquiry-based activity. This stage allows students to work collaboratively in teams, sharing and communicating their understanding through testing hypotheses, making predictions, and drawing conclusions. The major goal of *"the explain* stage" is to engineer student communication using individual and group presentations of what they have learned through the process of reflective thinking. The "e*xtend* stage" allows students to expand on the concepts, make connections and generalize the concepts. The purpose of the final "e" in the cycle — the "*evaluation* stage" — is to engineer on-going diagnostic process that allows both the teacher and the student to assess whether the desired level of understanding has been attained through implementation of well-designed rubrics, observation, interviews, peer-assessment, portfolios, and inquiry-based learning products/artifacts. This stage also addresses students' misconceptions and common mistakes.

Considering the key position of constructivism, the next question is how to become a constructivist teacher? First of all, a constructivist teacher is not just a teacher in a traditional sense; s/he is a facilitator, organizer, and coordinator of the problem-based student learning. Constructivist teacher by his/her very nature is a teacher-engineer. A constructivist teacher ensures favorable classroom environment for co-construction of students' new knowledge and understanding and encourages student initiative and collaboration. In turn, students become co-designers of the instructional process sharing the responsibilities for achieving learning objectives and outcomes with the teacher.

In lesson planning, a constructivist teacher prefers to consider real life problems including the context and data from practical situations and original sources. Moreover, a constructivist teacher provides opportunities for students to collect such data by observing real life situations, searching related information on the web, surveying participants, etc. Figuratively speaking, a constructivist teacher should engage both hands and brains of students.

Concepts, theories, algorithms, and theorems are abstractions that human beings create as a result of discovery. Theory is a retrospection. Accordingly, in the learning process an abstraction should be a destination rather than a starting point. Therefore, constructivism suggests focusing on exploration first, understanding main concepts and major ideas, and only then memorization of algorithms, rules, and theorems. Moreover, a constructivist teacher designs learning objectives using the cognitive terminology to emphasize understanding: classify, justify, analyze, synthesize, predict, evaluate, etc.

A constructivist teacher allows students to take over teaching of some fragments of the lesson, change the direction of the classroom discourse, offer ideas on improving teaching and learning. Obviously, constructivist teaching requires not only easy content handling but also profound pedagogical knowledge. Knowing-to-act at the moment (Mason and Spence, 1999) becomes a key ability for a constructivist teacher. This will allow a teacher to depart from the rigid structure of a lesson and transfer the "reins of power" of leading the lesson to students while focusing on achieving learning objectives of the lesson. A constructivist teacher never reports his/her view first. Instead s/he listens to students' viewpoints, explanations and justifications, involves students in a meaningful discussion, addresses and synthesizes diverse students' views and only then offers his/her view on the issue as one of the possible views among others.

A constructivist teacher encourages the development of students' critical thinking by considering opposing points of view, setting counterexamples, offering contradictions in order to promote a productive classroom discourse. S/he values good questions more than good answers. A constructivist teacher appreciates substantive, good, and 'smart' student questions. Moreover, a constructivist teacher often uses "waiting time" after asking a question: s/he leaves students enough time to think about an answer, make connections, and come up with analogies, images, and metaphors to provide a substantiated response. A constructivist teacher does not accept weak and short answers and always asks students to elaborate on their responses.

A constructivist teacher provokes students' curiosity by asking challenging questions and using heuristics to support student learning. S/he refrains from using low level teaching strategies to support student learning such as lecturing (providing information) and demonstration (showing how to do). A constructivist teacher would rather use advanced heuristics to support student learning such as demanding explanation and justification, providing hints (e.g., organize your data into a table), criticizing and posing counterexamples, sustaining high cognitive demand questioning (e.g., why, what if), and focusing on action (e.g., how did you do it).

Table 3 highlights differences between constructivist and traditional approaches to teaching and learning.

Table 3. Comparison of traditional and constructivist approaches to teaching and learning

	Traditional Approach	Constructivist Approach
Curriculum	Curriculum emphasis is on basic knowledge and skills	Curriculum emphasis is on major ideas and concepts
Teaching and Learning	Teaching and learning are predetermined by the strict implementation of the curriculum	Learning and teaching process is flexible with an opportunity to modify the curriculum
Resources	Teaching and learning is completely based on the recommended textbook	Textbook is not a dominant source of information
Student Positioning	Student is an object of the learning process. Knowledge is transmitted to a student	Student is a subject of the learning process. Knowledge is constructed by a student
Teacher Positioning	Teacher imposes knowledge, understanding and his/ her point of view on students	Teacher is a facilitator of student learning and understanding
Assessment of Learning Outcomes	Teacher evaluates the effectiveness of student learning by the number of correct answers	Teacher values student reasoning (even if it is not correct)
Criteria for Student Success	Test and exam results are the only source of information about the level of student knowledge and skills. Learning objectives, teaching, and assessment are usually considered in isolation	Student learning is assessed not only by test results but also by the efforts made by a student to achieve progress. Learning objectives, teaching, and assessment are closely connected

As any innovation, constructivism has some obvious flaws. At the current stage of its development, constructivism is more an educational philosophy than a learning technology, which causes some difficulties in the practical implementation of constructivism in the classroom. Some opponents accuse constructivism for undermining the foundations of organized teaching and learning. The main argument of opponents is fuzziness and lack of determination in teaching and learning (Anderson, Reder, & Simon, 1998).

Despite opponents' arguments, constructivism gets supporters among the teaching community, which is taking concrete steps to introduce the theory into practice. There is a shift from the old theories of behaviorism toward constructivism that takes place at different educational levels (e.g., schools, colleges, and universities). Most of the teacher training curricula are revised to include the principles of constructivism. Instead of studying the works of E. Thorndike, B. Skinner and other representatives of behaviorism, pre-service teachers study the works of J. Piaget, J. Dewey, L. Vygotsky and other constructivist scholars.

2.3. Constructionism

Constructionism is the theory of teaching, learning, and design advanced by Seymour Papert. Constructionism argues in favor of a more active participatory learning through social interaction and production of tangible learning outcomes. Learning, according to Papert, is "building relationships between old and new knowledge, in interactions with others, while creating artifacts of social relevance" (as cited by Kafai, 2006: 35).

Constructionism is closely related to the Piagetian constructivism theory. But they are not identical: constructivism places a primacy on the development of individual and isolated knowledge structures, whereas constructionism focuses on the connected nature of knowledge with its personal and social dimensions (Kafai, 2006: 36). In this sense, the Papertian constructionism shows resemblance with the Vygotskian social constructivism. In his original studies, Papert extensively used the programming language Logo to provide children with the opportunity to learn programming and to study Mathematics and Science through the manipulation of digital objects (e.g., Logo turtle), in interaction with others, and reflection on their own thinking and learning (e.g., metacognition).

In order to address the key ideas of constructionism, let us first consider the major distinction between two opposing approaches: innovative constructionism and traditional instructionism. Constructionism advances the idea of learning by constructing (e.g., knowledge, learning artifacts), whereas instructionism is associated with the traditional approach to teaching by transmitting knowledge. According to Mooney (2000), Piaget "claimed that children construct their own knowledge by giving meaning to people, places, and things in their world. He was fond of the expression "*construction is superior to instruction*" (p. 61). From this perspective, constructionism is strongly rooted in constructivism. In his pioneering publication "*Mindstorms*", Seymour Papert tried to define constructionism by contrast with constructivism:

"Constructionism — the N word as opposed to the V word — shares constructivism`s connotation to learning as building knowledge structures irrespective of the circumstances of learning. It then adds the idea that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity whether it is a sand castle on the beach or a theory of the universe" (Papert, 1993: 1).

According to the definition, active engagement in learning through construction of 'a public entity' is the central aspect of constructionism. Moreover, the context and environment are critical in stimulating learning and construction of knowledge. Technology plays a key role in the constructionist classroom because it enables students to create 'public entities' and develop both cognitive and affective skills while acting as the agents of learning. According to constructionism, manipulation of objects facilitates the connection between the old knowledge and a new concept.

The Papertian constructionism builds on the similar constructivist idea and proposes the term "objects-to-think-with". This process of mental identification with the object supports the mechanism of appropriation and is called *syntonic* learning. Papert used the Logo Microworlds as an example of the computer-based 'objects-to-think-with' approach that provided students with the opportunity to construct artifacts through designing their own programs and construct their knowledge and understanding at the same time.

To clarify the difference between constructivism and constructionism, Kafai explains that, though both theories involve the mechanisms of assimilation and accommodation, constructionism goes beyond these essentially cognitive processes placing high emphasis on *appropriation,* a social interactive process, which suggests that "learners make knowledge their own and begin to identify with it" (Kafai, 2006: 39). In other words, appropriation is an essential final stage of the learning process: students are expected to construct knowledge and design their own artifacts by applying the concepts they have gained to new situations.

As the emphasis on appropriation is one of the main distinctions between constructivism and constructionism, it will be discussed further after a brief explanation of assimilation and accommodation mechanisms. Assimilation and accommodation are complementary processes that can be best understood by the comparison with Donovan and Bransford principles of learning (2005). Assimilation corresponds to the first principle of learning — it involves drawing on prior knowledge (e.g., already existent cognitive schema) to understand the new information. Accommodation refers to the second principle of learning presented by Donovan and Bransford (2005) — the relation between factual knowledge and conceptual framework to support understanding. In other words, conceptual understanding helps to create a new schema that helps to accommodate the factual knowledge acquired. Appropriation is an inherent aspect of learning highly emphasized in constructionist classrooms. As appropriation implies ownership of knowledge, it requires that a learner develops strong self-monitoring and metacognitive strategies, which Donovan and Bransford (2005) identify as the third principle of learning. Metacognition is critical in the knowledge appropriation because it "includes an awareness of the need to ask how new knowledge relates to or challenges what one already knows — questions that stimulate additional inquiry that helps guide further learning" (Donovan & Bransford, 2005: 11). Here lies the value of technology as "objects-to-think-with", which helps students to develop cognitive and affective skills, as well as metacognitive competence as they engage in both individual and collective activities involving designing 'public entity' (e.g., a computer program) and constructing understanding.

Kafai's argument (2006) that technology facilitates the knowledge appropriation in constructionist classes seems very plausible as she provides examples of the research projects that corroborate this point. In one of the studies, older students were required to design instructional software to teach fractions to younger learners. The project provided substantial evidence to conclude that younger students highly benefited from the software, whereas student-designers greatly improved their programming skills and conceptual knowledge of fractions in addition to developing metacognitive competence. Another revealing research evidence substantiating the cognitive, metacognitive, and affective gains of using technology as facilitator of learning is a study on the involvement of ten-year-old students in designing and programming their own computer games. The project provided students with an opportunity to design games according to their interests, which is very important as the knowledge appropriation process requires learners to be engaged in activities that are interesting, relevant, and meaningful to them (Donovan & Bransford, 2005).

The Papertian constructionism also distinguishes from the Piagetian constructivism with regard to cognitive development. Constructivism "places a primacy on the development of individual and isolated knowledge structures" (Kafai, 2006: 36), whereas constructionism emphasizes the role of social interactions in influencing learning. This is a bridging point that constructionism builds between Piagetian and Vygotskian views on constructivism. Vygotsky (1978) considered learning as a result of collaboration and socialization. Furthermore, Vygotsky emphasized the role of language in facilitating the learning process. Vygotsky (1978) also stressed the role of the teacher as a dynamic and effective contributor to the learning process by providing the needed scaffolding to learners until they are able to execute the task independently.

Yet, the apparent major distinction between constructionism and constructivism, regardless of its Piagetian or Vygotskian interpretation, is the emphasis that constructionism places on the production of an artifact that can be shared and reflected upon with others in addition to being personally meaningful. Perhaps, this distinction may have resulted from the evolution of technologies, which lead to advancing constructivism toward constructionism.

Papert also strongly emphasized the role of learning culture in knowledge co-construction and claimed that "…this suggests a strategy to facilitate learning by improving the connectivity in the learning environment, by actions on cultures rather than on individuals" (cited by Kafai, 2006: 39). To emphasize the influential role of learning cultures, Papert describes how learning is facilitated among the members of the Brazilian samba schools where the group participants of different age learn from each other. Another difference refers to the equal value of concrete and abstract thinking in constructionism. Papert and Turkle (1992) discovered that "the top-down or planning approach was not always superior to a more improvised, more bricoleur-like approach" (p. 30). Table 4 summarizes major differences between the Piagetian constructivism and the Papertian constructionism via multiple lenses.

Regardless of the differences between constructivism and constructionism as depicted by Table 4, these theories play a significant role in providing a solid foundation for framing and interpreting emerging learning phenomenon in the digital age.

Table 4. Comparison of the Piagetian Constructivism and the Papertian Constructionism

	The Piagetian Constructivism	The Papertian Constructionism
Theoretical focus	The theory of knowledge development	The theory of learning and teaching
Primary dimensions in knowledge development	Places primacy on the development of individual and isolated knowledge structures	Focuses on the connected nature of knowledge with its personal and social dimensions
View on learning	Views learning as building relationship between old and new knowledge	Views learning as building relationship between old and new knowledge in interaction with others
View on knowledge construction	Views knowledge construction as an individual act	Articulates a more distributed view of knowledge construction
Aspects of learning	Concerned primarily with cognitive aspect of learning	Concerned with combination of cognitive and emotional aspects of learning to address "knowledge as desire" phenomenon
Primary learning outcome	Building a cognitive schema	Creating an artifact of social relevance
Cognitive mechanism of knowledge construction	Builds on the mechanisms of assimilation and accommodation	Extends the mechanisms of assimilation and accommodation to the process of appropriation
Role of technology in knowledge construction	Technology is not a primary focus in knowledge construction	Knowledge co-construction and appropriation is facilitated by interactive activities involving technology
Relationship between concrete and abstract	Distinguishes between concrete and abstract thinking and considers the latter as more advanced	Equally valuing concrete and abstract: "concrete thought could be just as advanced as abstract thought" (Kafai, 2006)
Learning culture	Concerned primarily with individual learning and development irrespective of other circumstances of learning	Emphasizes the importance of learning cultures with focus on apprenticeship models
Learning environment	Authentic learning is not a primary consideration	Values learning environments, which promote authentic and syntonic (e.g., responsive and adaptive) knowledge construction

Connectivism

With growing ICT integration in teaching and learning, there are new theoretical models branching out of constructivism. One of these emerging branches is connectivism.

Downes (2007) identifies the core proposition shared between social constructivism and connectivism as the knowledge 'not being acquired, as though it were a thing'. Connectivism is a theoretical framework for understanding learning through the process of connecting to and feeding information into a learning community (Kop & Hill, 2008). Siemens (2005) further clarifies, "A community is the clustering of similar areas of interest that allows for interaction, sharing, dialoguing, and thinking together." Within the connectivist framework, "a learning community is described as a node, which is always a part of a larger network. Nodes arise out of the connection points that are found on a network" (Kop & Hill, 2008) and knowledge is distributed across the network and "rest in diversity of opinions" (Siemens, 2008). This leads connectivists to propose the following definition of learning: "learning is the network" and, therefore, learning can reside outside of ourselves (within a network or a database).

Downes and Siemens attempt to locate the construction of distributed-knowledge among other epistemological frameworks such as objectivism, pragmatism, and interpretivism (Driscoll, 2000). Objectivism claims that reality is external to mind, and knowledge is experientially acquired whereas pragmatism positions knowledge as a negotiation between reflection and experience. Interpretivism persuades that knowledge is an internal construction through socialization and cultural cues (Driscoll, 2000). Siemens (2008) further argues that "the concept of emergent, connected, and adaptive knowledge provides the epistemological framework for connectivism…" (p. 10) and suggests the following alignment between epistemologies and learning theories: objectivism => behaviorism; pragmatism => cognitivism; interpretivism => constructivism; distributed-knowledge => connectivism.

As any emerging framework, connectivism has its weak points that are criticized by opponents. Thus, Kerr (2007) suggests that the basic ideas of connectivism had already been proposed by Clark (1997) in his theory of embodied active cognition built on the Papert's constructionism. Verhagen (2006) cannot distil any new principles from connectivism that are not already present in other existing learning theories. Critics also argue that recent widespread attention to the work of connectivism is mainly due to the high visibility of networks in the digital age. Moreover, critics including the author of the book are not convinced that learning can reside in non-human appliances (Kop & Hill, 2008).

2.4. Social Constructivism in Action

Implementation of the social constructivist approach in the classroom requires knowledge of specific teaching methods and techniques. One of the wide accepted methods is the cooperative learning. When the author asked his graduate students to define what was a cooperative learning one of them wrote "… social constructivism in action".

There is a sustained interest to use cooperative learning in both face-to-face and online classes. Modifications of cooperative learning include but are not limited to the team-based learning, collaborative learning, learning in small groups. In this section we will focus on a number of specific issues related to the implementation of cooperative learning: what constitutes a cooperative learning, what is an optimal size of a small group in cooperative learning, how to implement specific cooperative learning techniques in the classroom.

The major research in cooperative learning took place in the 1980-ies. During this period, a number of studies on the benefits of learning in small groups were conducted by N. Davidson (1980), N. Webb (1982) and cooperative learning — by D. Johnson & R. Johnson (1999), R. Slavin (1983), and others.

Let us start the review of cooperative learning by asking the question: is any learning in small groups cooperative? The quick answer is "no". According to the above mentioned studies, in order to ensure that learning in small groups is cooperative, it should meet the following basic requirements: the majority of classroom and extra-curricular activities should be carried out in small groups (3-5 people in each group); each small group must possess a collective spirit — the team spirit; each team member should be responsible for him/herself, for others and for the members of the team as a whole; it is preferable that student's membership in a team is stable and permanent within the class and across different classes; and, last but not least, the collective student work should be considered as student achievement in the course of progress assessment (Davidson, 1990; Johnson & Johnson, 1999).

The theoretical basis of cooperative learning pedagogy is grounded in the works of L. Vygotsky and other scholars who emphasized the critical role of social interaction and interpersonal communication in learning and intellectual development. Studies (Sharan, 1990; Reynolds, 1995) showed that communication in the process of cooperative learning had a positive effect on the development of students' language, thinking and intelligence. Moreover, well-organized cooperative learning contributed to higher learning outcomes than traditional forms of teaching. Even the most dynamic and informative lecture, in general, was less efficient than learning in small groups with a skillfully constructed communication among students.

Let us consider the following main practical issues related to the implementation of cooperative learning in the learning process: formation of small groups; cooperative learning techniques; methods of cooperative learning; assessment of group achievement.

The basic starting positions in planning cooperative learning are related to the composition, size, structure and the "lifespan" of a small group. First, the principle of heterogeneity (diversity) in the formation of small groups should be taken into account. Studies show that homogeneous (uniform in terms of learning) groups are not effective: the strong groups become stronger and the weak — even weaker. On the other hand, studies show that the heterogeneous composition of small groups significantly improves learning and achievement of weak and mid-performing students and, at the same time, stimulates the academic progress of advanced students. In addition, small groups should be formed using the following criteria: variety of educational interests, social and psychological characteristics and psychological compatibility of group members; diversity of learning styles and preferences, etc.

The second issue is related to defining the optimal size of a small group. Some educators feel that the most appropriate size of a small group is three students per group. Others suggest five students in a group. The option — two students per group is not considered as a learning team. Observations show that the optimal size of a small group — four students per group. It is also reported that this size of a small group has the highest degree of efficiency and productivity, and is the most suitable for intra-group communication (Reynolds, 1995). There are also some other advantages for this particular composition: it could be easily rearranged into two subgroups of two students (it is convenient to work in pairs). This is also the most ideal combination for heterogeneity in terms of academic performance (one strong, two medium and one weak student per group) and in terms of gender (two males and two females).

It should be noted that the formation of a small group is rather complicated process if one does not carefully consider the factor of group dynamics. If the group dynamics is not addressed, a group can work productively for a while and then quickly disintegrate. On the other hand, a carefully formed group will consistently and effectively operate over a long period of time.

The principles of cooperative learning should be applied at various stages of a lesson plan: starting at the exploration stage and ending at the evaluation stage. The main goal of cooperative learning at the exploration stage is to link the prior collective knowledge of group members to the new knowledge through collaborative project. Let us consider an example of a collaborative project for the topic "Trigonometric relationships between the sine and cosine of the numeric argument." Each group receives a description of the project and graphing calculators. The groups are asked to record their observations of function behavior and write conclusions for further discussion in the classroom.

- Using a graphing calculator, plot graphs of functions f, g and $f + g$, where $f(x)=\sin^2 x$ and $g(x)=\cos^2 x$. Observe the behavior of the function $f + g$.
- Plot graphs of the following functions: $f(x)=\cos x$ and $g(x)=\sin(x+c)$. By varying the parameter c describe the behavior of the graphs of two functions. Record the values of the parameter, at which the graphs of these functions are the same.
- Conduct the same observation for the following functions: $f(x)=\sin x$ and $g(x)=\cos(x+c)$.
- Plot graphs of the following functions: $f(x)=\cos 2x$ and $g(x)=\sin^2 x$. How can a graph of the function $g(x)$ be transformed to get the graph of the function $f(x)$?

Students work in small groups for 10-15 minutes constructing the appropriate graphs of functions, discussing the results, asking questions, clarifying obscure points in assigned tasks, formulating main conclusions from observations, recording findings on the answer sheet, etc. Each group will be asked to present their major finding. At the same time, each team member must be willing to speak on behalf of the team. During the group work the teacher monitors the work of teams, asks guiding questions to clarify certain points, provides recommendations to better articulate findings, etc. In other words, the teacher coordinates and directs the group work.

During the group presentations, the teacher selects one of the groups to share its results. A speaker presents the findings on behalf of the group. For each session the group appoints its speaker. Each member of the group should get an opportunity to be a speaker. At the same time, the group might decide to present as a whole group where one of the group members demonstrates the graphics, another member comments on the findings for the first task, the next member reports results for the second task, etc. While the first group presents, the members of other teams listen to the presentation, ask questions, offer their findings and conclusions if they disagree with the presented results, and express support if they have the same results. Moreover, the members of other teams and the teacher have the right to address questions or comments to any member of the presenting team. Therefore, it is crucial that each member of the team is able to explain any task and answer questions s/he is asked on behalf of the whole team. During cooperative learning, the teacher and the team members need to maintain friendly atmosphere in the process of discussion with the elements of constructive criticism. The teacher acts as a discussant following the rules and moderating the question and answer session without imposing his/her point of view. At the end of the discussion, the teacher briefly summarizes the results obtained by the groups, records major findings on each task, analyzes typical errors and closes the discussion.

At the stage of learning new material the main purpose of the group work is to provide formal proof for the empirical findings of the group obtained at the exploration stage. The sequence of the group work at this stage is similar to the exploration stage of cooperative learning. The third stage of the lesson is application of the newly learned material: at this stage groups can work collaboratively on assigned problems. Also, the teacher might administer a test to monitor and evaluate individual students' progress. Moreover, an additional project could be assigned as a collective or individual homework. Thus, the teacher combines group and individual assignments during the cooperating learning. If the homework project is assigned as a group work, the team determines the scope and sequence of work as well as the distribution of tasks between the team members. After doing each part of the distributed homework individually, the group meets to discuss solutions, during which each team member has an opportunity to understand solutions presented by the other members through asking questions, discussing results, and correcting solutions if necessary. An important requirement for the group homework is that each member of the group should know how to solve every problem in the assigned homework project and be able to present and justify the solution on behalf of the group. Performance of each group member will impact the final group's grade for the homework project. In the process of grading the group homework, the teacher has a right to selectively invite individual team members for questions and comments on the solutions for specific tasks encouraging each member to be responsible for the results of the entire group, which motivates students to work hard on the group homework projects.

There are diverse methods that can be used in cooperative learning (Webb,1992). Let us consider some of the cooperative learning methods.

Jigsaw method is implemented through the following sequence of steps: students are divided into teams of four students and the course material is divided into four parts. Each student is assigned to study one of the parts. Then, members of different teams who have studied the same part are brought together for 10-15 minutes to discuss the new material. After the discussion, the students return to their teams and each student of the team in turn explains the content of the assigned part to the rest of the team. The student knowledge and understanding of the new material is assessed by individual test or quiz. The winner is the team that gains the highest cumulative team score on the test. The main feature of this method is interdependence of the team members in learning: the team success depends on the individual work of each team member and on the individual contribution of each member to the collective learning and performance.

Achievement team. This method is implemented as follows: lecture — group work with the text — individual self-study. At the beginning of each lesson, the teacher delivers a brief lecture to provide an overview of the new material with an emphasis on the main points, which later will be used to solve problems assigned to each group. The lecture should be sufficiently broad in content and practical application. Next, students work in teams on lecture notes and help each other to understand its content. While working in groups, the students are involved in the discussion to clarify the main points of the lecture. The students are allowed to ask the teacher only when none of the team members can answer a question. After the group work is done, the students carry out an individual assignment. At this stage, each team member is working on his/her own without interaction with the other team members. The main focus of this method is on individual student achievement that will be added up to the team score. The importance of each student effort is enhanced through the following arrangement: an individual student score counts if it is above the student's average score for his/her previous work. The team receiving the highest score is the winner.

Team contest. The main characteristic of this method is that students at the same level of academic achievement compete in the team contest. As a rule, this kind of the team contest takes place once a week after the major topic was studied. Students from all teams are divided into groups according to the level of educational achievements: strong students form the first group, the mid-level students — the second group, and the low-achieving students — the third group. Then each group receives about thirty cards placed on the table in random order (questions down). Each student from the first group selects a card and answers the question written on it. The contest can be conducted either in oral or in written form. Other students from the same group evaluate the answer, for example, using alternative scale: correct (1 point) and incorrect (0 points). In case of a dispute, the students may ask the teacher to be an umpire. In average, each student answers three questions. Thus, for this contest the teacher needs to prepare about 90 cards with tasks for three levels of difficulty. After the contest in groups, the students return to their teams and add up the obtained team scores. The team with the highest score is a winner.

Team-based individual learning. The essence of this method is to provide small groups an opportunity to move through the curriculum at their own pace. Students work in small groups on individually assigned tasks based on the previously learned material and can access each other for advice, help, and assistance. The students are also allowed to work with each other in a team to address misconceptions and correct mistakes. The teacher oversees the group work and explains the new material to those groups who first completed the work on individual tasks. Individual tasks are evaluated by students from different groups appointed as teaching assistants. The teaching assistants are provided with answer sheets which help them to timely assess individual student's performance. At the same time, the teacher has an opportunity to explain new material to each small group. Individual scores are added up to compose a team score at the end of each unit (week). It is clear that the implementation of this method requires careful design of individual assignments and tests for each unit on the teacher side. In addition, the teacher must skillfully allocate study time to work on the new material with each group separately.

Cooperative team learning method requires continious mutual understanding and support from team members through peer-tutoring and peer-assessment. This method can be used in various forms of study groups: formal (formed according to criteria specific for a particular learning task), informal (formed on the basis of sympathy or friendship), and basic (formed to address the long-term educational goals).

Team project. The key feature of this method is that the course material is divided among teams, so that by the end of the term students learn the entire course. Each team is assigned a special topic. Teams work to prepare the group report on a topic and present it to the class. Within each team, the topic is divided into units. Each student is assigned a unit to independently work on. The student prepares his/her part of the report, submits it to the group, and then, the team compiles the group report based on the individual units submitted by the team members. Each team receives a group grade for the project.

Coop-coop method. This method is very similar to the team project method with the only difference that, in addition to delivering part of the report to the team, each member makes a mini-presentation. After the final team report is compiled, the group speaker makes a presentation for the team members and then — for the whole class. In addition to the group effort each student takes a test. The final student grade consists of the group grade on the project and the individual grade on the test.

Experiment in cooperative learning. The main purpose of this method is to transform a randomly composed small group into a cooperative learning team. That is if a class, for example, consists of 32 students, every student gets a randomly assigned number from 1 to 8. Thus, the first group is formed by four students, who were randomly assigned as the "first", the second group — by students randomly assigned as the "second", and so forth. The total number of the groups is eight. The main goal of this method is, regardless of random formation of groups, to promote a friendly and productive learning environment in each group. In other words, a small group will become a cooperative learning team by the end of the term. In order to achieve this goal, small groups are encouraged to start by identifying common interests among group members, building team spirit, etc. This method is particularly recommended for beginning teachers to learn team building skills in cooperative learning.

Inquiry-based team learning. This method is aimed to build teams of students for research, solve practical problems and/or to implement applied projects at a high level of complexity and challenge. This method requires a certain level of independency for each group. Therefore, groups may be formed using arbitrary (often informal) criteria. The main goal set for each group is to conduct a mini-research that requires creative approach to identify a problem, to formulate a hypothesis, to gather empirical data, to conduct statistical analysis, to write a research report, and finally, to defend the research results before a special advisory council consisting of teachers of different disciplines, parents, and students.

The above methods do not exhaust the whole arsenal of cooperative learning techniques. Implementation of these methods illustrates a wide range of practical applications of the social constructivist approach in the classroom. Methods can be combined and used in conjunction with the conventional teaching methods. Furthermore, cooperative learning is an open and dynamic system that is continuously improving by teacher initiative and creativity. The cooperative learning methods and techniques discussed above could be easily modified to be used in online teaching.

Last but not least, the group assessment should be clearly defined in cooperative learning. Studies show that the group grades should not exceed 50% of the total grade for each individual student (Davidson, 1980). One should be careful to ensure that the group assessment does not significantly reduce the strong individual student performance and, at the same time, does not increase the unjustified weak individual student achievement. Therefore, it is critical to clearly assign grade weights for every group and individual assignment. Implementation of cooperative learning requires special training of teachers, in particular, to prepare teachers to overcome challenges that can arise in the real classroom. When arranging cooperative learning teachers should also be prepared to resolve some irregularities and constraints with regard to the task assignment and completion. It might happen that individual members, who are not supportive of the group work, are lagging behind in completing homework projects, etc. One can expect difficulties related to group dynamics when high achieving students dominate group discussions, refuse to provide assistance to other group members. The so-called growth problems or difficulties associated with group dynamics, formation and development of the group as a team. In each case, a teacher needs to patiently explain the principles of cooperative learning, hold informal meetings with the groups facing problems, emphasize positive qualities of the group and its individual members, and support psychological compatibility among group members. It is also important to emphasize the ability to work in teams.

2.5. Learning Culture and Multiple Intelligences

Learning culture embraces diversity and its inclusiveness. The theory of multiple intelligences developed by H. Gardner (1983, 1993, 2000, 2004) provides an opportunity to expand the humanistic dimension of learning culture.

In this section we describe the key ideas of Gardner's theory. In 1904, by the request of the Minister of Education of France psychologist Alfred Binet and his colleagues designed an instrument to measure students' intelligence. A little later, the instrument was imported to the United States and became famous as an Intelligence Quotient (IQ) test. Almost 80 years later, the Harvard psychologist Howard Gardner challenged the theory of IQ. He proposed the Theory of Multiple Intelligences (TMI), which is diametrically opposite to the IQ-theory. The key conceptual ideas of Gardner's theory are:

- Intelligence cannot be measured in the laboratory by any test (including the IQ test);
- One cannot justify racial, ethnic, and religious differences based on the results of any intelligence test;
- Human intelligence is multiple.

The last attempt to defend the IQ theory after the crushing blows of the TMI was the book "The Bell Curve" (Herrnstein and Murray, 1994). However, this attempt was unsuccessful. Currently, the Theory of Multiple Intelligences is well respected among scholars and practitioners as an approach to recognize and support the diversity in human learning and development.

Gardner's theory is based on the fundamental psychological and neurophysiological studies carried by his predecessors. Let us describe the most important points. In 1981, Roger Sperry won the Nobel Prize for his work on the neurophysiological mechanisms of specialization of the left and right hemispheres of the human brain in processing information. Sperry found that the left hemisphere processed information linearly, consistently, and in parts, while the right hemisphere — simultaneously, in parallel, and holistically. Paul MacLean, Head of the Laboratory of Brain Evolution at the National Institute of Brain (Washington, DC), found that the human brain consisted of three layers that were laminated to one another as a person grows and passes to a higher level of thinking and mental development. Karl Pribram from Stanford University proposed a new theory of human brain functioning as a hologram: the information had been imprinted into the human brain, hence each individual piece of information also belonged to its integral structure. That is why by recalling a single episode one can reproduce the whole picture of a past event.

On the contrary, the advocates of the IQ theory believe that the intelligence of a person is predetermined, fixed, and static. That is to say, intelligence is something that is given to a human from birth and it does not change throughout human life.

However, neuropsychological studies suggest the opposite: that intelligence can be changed and developed throughout life. Moreover, Gardner argues that human intelligence can be improved and developed in multiple directions (Gardner, 1993).

Gardner defines intelligence as an ability for nonstandard problem-solving, generating new ideas, and creating products and services that have high social and cultural value. Originally[2], the following types of intelligences were mentioned: linguistic intelligence, logical-mathematical intelligence, spatial intelligence, bodily-kinesthetic intelligence, musical intelligence, interpersonal intelligence, and intrapersonal intelligence. Let us briefly examine each of the above types of intelligence.

Linguistic intelligence is responsible for the diversity of spoken and written languages including knowledge and skills in grammar, reading, writing, poetry, and even humor. This type of intelligence is inherent to writers, poets, storytellers, etc. Logical-mathematical intelligence is associated with the so-called scientific thinking, that is, an ability of inductive and deductive reasoning, logical thinking, an ability to handle abstractions, symbols and numbers, an ability to establish cause-effect relationships, reveal patterns, an ability to connect concrete and general, etc. This type of intelligence is clearly manifested in scientists, computer programmers, accountants, lawyers, bankers, and, of course, mathematicians. Spatial intelligence is related to such human abilities as painting, sculpture, design, navigation, architecture, and, strangely enough, chess-playing. This type of intelligence involves a highly developed ability for imaginative thinking and, as a rule, is inherent to professionals in the field of architecture and art, cartography, engineering, etc. Bodily-kinesthetic intelligence reflects the ability of a person to perform creative expression of emotion, strength and beauty of using plastic movements of certain muscles and body as a whole. Accordingly, this type of intelligence is particularly pronounced among dancers, athletes, acrobats, skaters, etc. Musical intelligence includes but is not limited to the ability to recognize and use voice and rhythm, sensitivity to sound and tone, highly developed musical audition, and an ability to play musical instruments. This intelligence, as the previous one, is very close to the human nature: many of us love to sing or whistle favorite tunes while performing body movements (dancing, walking, playing, etc.). Obviously, the most vivid expression of this type of intelligence is manifested among composers, singers, performers, music teachers, etc. Interpersonal intelligence involves highly developed communication skills, an ability to work in a team, an ability to sustain close contact and communication with the audience, an ability of psychological understanding of another person. This type of intelligence is inherent to therapists, religious leaders, politicians, managers, etc. Intrapersonal intelligence involves, above all, the knowledge of the inner mechanisms of human mental activities at the level of feelings, emotions, self-awareness, intuition, etc. It also includes an ability to reflective thinking, in-depth analysis, and internal dialogue. This intelligence can be found in thinkers, philosophers, spiritual leaders, etc.

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[2] Later, two types of intelligences were added to the list: naturalistic intelligence and existential intelligence (Gardner, 2004).

There are four key concepts of TMI. First, each person has a natural aptitude to a particular type of intellectual activity. It is quite natural that different people have different abilities that are developed to a varying degree of mastery: some have universal intellectual abilities and others show their intelligence in a specific area. For example, the famous German writer J. W. von Goethe was a statesman, a scientist, and a philosopher. On the other hand, there are many examples of brilliant people who have had extraordinary intellectual ability in a single area, for example, Carl Gauss — in mathematics, Bobby Fischer — in chess, etc. At the same time, many people are in the middle between the above extreme cases of human intelligence.

Second, the majority of people are capable of developing any type of intelligence to an adequate level of competence. In other words, we cannot assert that this or that person has no aptitude for mathematics, music, literature or art; they just had not developed it properly. In the same way, every student has a potential to learn any school subject, such as mathematics, if necessary conditions have been created.

Third, different types of intelligences can closely interact and influence each other's development. There are cases when student's involvement in music contributed to the development of his/her mathematical abilities. That is why it is important to engage students in multiple learning activities through games, plays, music, and sports.

Fourth, there are different ways to develop different intelligences. Thus, in order to be a good storyteller, it is not necessary to know how to read and write. However, this does not mean that we should not learn to read and write in order to be eloquent. This example only underscores the fact that intellectual ability can be developed in a variety of ways.

The key positions listed above provide an opportunity to consider TMI as an open system. It can be extended with new varieties of intelligence as it happened later with naturalistic and existential intelligences (Gardner, 2004). The only requirement is the need for scientific evidence of the existence of a new type of intelligence (Armstrong, 1994; Campbell, Campbell and Dickinson, 1994; Lazear, 1999).

How can multiple intelligences be developed? The development of intelligence requires the use of appropriate teaching methods and learning activities. For example, in the development of the linguistic intelligence the most favorable learning activities can be reading, working with the dictionary, language development, keeping a diary, writing narratives and essays, poetry writing, discussions and debates, using humor in the classroom, storytelling, etc. For the development of the logical-mathematical intelligence it is recommended to use a variety of challenging problems, logical games and puzzles, fallacies and paradoxes, etc. Unfortunately, many teachers consider the logical-mathematical intelligence only as an ability to master narrow subject knowledge and skills in mathematics and logic. It happens that a student has a good command of mathematics at school, but his/her logical-mathematical intelligence is not sufficiently developed. That is why, for the development of this type of intelligence it is necessary to include non-routine problems, discovery activities and student involvement in research.

The development of the spatial intelligence involves such activities as drawing, sculpting, creating spatial models of objects, initiation of active imagination, fantasy, use of visual aids, videos, etc. Bodily-kinesthetic intelligence can be developed through implementation of role-playing, dramatization, dancing, using body language, sports and other activities. Musical intelligence is most fully manifested in the educational gaming activities related to playing music, singing, using audio materials, natural sounds, etc. Interpersonal intelligence is well developed in the process of learning in small groups, collaborative teams, performing group projects, peer assessment, etc. The development of intrapersonal intelligence involves the use of individualized learning techniques, development of students' self-monitoring skills, use of mediation strategies, reflective thinking and metacognition, development of students' intuition.

Obviously, the development of any specific type of intelligence should not be associated with a specific discipline only, for example, the development of the linguistic intelligence — with languages and literature, the logical-mathematical — with mathematics, the spatial — with fine arts, and so on. On the contrary, it is necessary to design the learning process in a way that each discipline is involved in the development of various types of intelligence. An example of the unit plan on "Discovery and Invention" aimed at the development of different types of intelligence is presented in the matrix below (Table 5).

Table 5. Using multiple intelligences across disciplines

Intelligence	Disciplines		
	Mathematics and Science	Language Arts	Social Studies
Linguistic	Reading literature on the history of inventions and discoveries in mathematics. Discussing scientific principles of a particular discovery or invention	Searching the Internet and reading encyclopedia of discoveries and inventions. Writing a short story "What would you like to invent?"	Writing an essay on social and historical conditions, under which a particular scientific discovery was conducted
Logical-mathematical	Studying a theorem or formula used in the construction of a particular invention. Building a hypothesis for a new invention	Reading a popular book on the role of mathematics in scientific discoveries. Describing the practical situation that would follow a particular discovery	Building a chronological timeline of the most important discoveries in mathematics
Spatial	Plotting a graph or a geometric model of the object included in a particular invention. Drawing a sketch of invented mechanism	Reading an illustrated book about the discoveries in culture, art, and design. Compiling a dictionary of terms of discoveries in culture, art, and architecture	Decorating a newspaper devoted to a discovery in social sciences
Bodily-kinesthetic	Inventing a device for measuring a parameter of a muscle (strength, endurance, etc.). Constructing a physical model of a technical invention	Reading operating instructions for a specific invention. Writing a user guide for self-designed technical model	Creating and executing a play or "historical drama of ideas and people" about a particular discovery or invention

Intelligence	Disciplines		
	Mathematics and Science	**Language Arts**	**Social Studies**
Musical	Studying mathematical foundations of designing new musical instruments. Learning about scientific principles of electronic records and digital music production	Reading biography of the inventor of a musical instrument. Writing lyrics to the music dedicated to discoverers and inventors	Listening music records from a variety of historical eras
Interpersonal	Small group study of mathematical foundation of a discovery. Group involvement in the debate on scientific discoveries	Reading and discussing an article on international scientific cooperation. Conducting a contest of the best collective play about scientific discovery	Group discussion on cultural-historical occurrence of scientific discoveries
Intrapersonal	Independently solving math problems drawn from the results of a particular discovery. Developing a program for self-study of scientific principles of a specific invention	Reading a book about the author of a discovery or invention. Writing your own autobiography as a famous inventor	Reflection on the topic “What would you do if you invent a time machine?”

What are the advantages and disadvantages of the Theory of Multiple Intelligences? Among other advantages, the major strong points of TMI are as follows: new understanding and definition of intelligence; ample opportunities to develop students’ skills with different types of intelligences; recognizing multiple forms of human intelligence in a variety of fields: literature, science, art, music, sports, politics, religion, etc.; democratic nature of the theory: each type of intelligence has the right to be supported and developed; unique opportunities to enrich the learning process through different types of intellectual activities. Disadvantages of the theory are primarily related to the blurring boundaries of its interpretation and application. For example, the distinction between the notions of “talent” and “intelligence” is not clearly defined. What are the boundaries of TMI in terms of incorporating new types of intelligence? For example, could extraordinary culinary skills be considered a manifestation of intelligence? The limitations of TMI have not been explicitly described. In general, the Theory of Multiple Intelligences developed by Howard Gardner is, without a doubt, an innovative contribution to the science of learning that has a great potential to improve student learning in the digital age.

Chapter 3

The Engineering of Learning Toolkit

This chapter addresses the following issues:

- analysis and design of learning objectives;
- cognitive tutoring, representations, and new literacies;
- research-based strategies in engineering of learning;
- assessment of learning outcomes.

3.1. Design of Learning Objectives, Tasks and Didactical Situations

The design of learning objectives is one of the key elements in the engineering of learning toolbox. Effective teaching and learning equally depends on each element of the triad: objective — content — outcome.

The classical fundamental concept in this field is the taxonomy of educational objectives developed by B. Bloom and his colleagues (1956). Later, Bloom's taxonomy was modified in the works of W. Gerlach and A. Sullivan (1967), A. De Block (1975), J. Guilford (1967), R. Gagne, M. Merrill (1964, 1971), R. Marzano and J. Kendall (2006), and others.

The term 'taxonomy' (from the Ancient Greek "taxis" — arrangement and "nomia" — method) means organization, arrangement, and classification of objects according to a certain method, criteria and principle, setting their hierarchy (a sequence of levels in a particular structure).

Bloom's Taxonomy. Despite the fact that Bloom's taxonomy was developed over 50 years ago, however, it is still one of the most popular classifications of educational objectives among both scholars and education practitioners. This taxonomy is the most complete classification of educational objectives, it covers a variety of learning activities of students in different domains: cognitive, affective, and psychomotor. In this section, we will mainly focus on the cognitive domain. Bloom's taxonomy is based on the following principles: taxonomy should rely upon the theory of goal-setting as well as be instrumental for teaching practice; it should be based on the advances of modern psychology; it should be logically complete and have its internal structure; the hierarchy of objectives does not imply the hierarchy of values. Based on these principles, the taxonomy of educational objectives in the cognitive domain includes the following six levels: (1) knowledge, (2) understanding, (3) application, (4) analysis, (5) synthesis, and (6) evaluation.

The level of knowledge involves the development of educational objectives aimed at memorization, recognition and reproduction of basic definitions, rules, algorithms, and procedures. The objectives at this level include the following categories: specific knowledge (dates, facts, numbers, terms, names, etc.); procedural knowledge (criteria, directions, classes, rules, etc.); abstract knowledge (principles, axioms, theorems, theories, etc.). The level of understanding includes learning objectives in the following categories: translation (e.g., an ability to translate a task from everyday language to the language of mathematics); interpretation (e.g., an ability to explain a mathematical solution using everyday language); extrapolation (e.g., an ability to transpose knowledge to a similar situation). The level of application involves the development of students' applied skills to use the knowledge in practical situations. This level may also be subdivided into the categories: application of concepts; application of methods and algorithms; and application of theories. The level of analysis contains learning objectives in the following categories: analysis of elements (e.g., separation of a whole into parts); analysis of relationships (e.g., establishing links between elements); analysis of principles (e.g., sequencing and placing elements in a certain order). The level of synthesis consists of the following categories: synthesis of ideas (e.g., search for ideas to tackle a problem); synthesis of procedures (e.g., development of a plan, a sequence of steps to solve a problem); synthesis of structures (e.g., design of principles for groups, sets, theories). The level of evaluation contains the following categories: evaluation of abilities that relate to the internal knowledge and belief system (argumentation, logic, constructive critique, etc.); evaluation of abilities that relate to the use of external criteria (standards, rules, regulations, etc.).

Modifications of Bloom's taxonomy. Along with advantages for designing learning objectives, Bloom's taxonomy has some drawbacks. Thus, though the systematization of objectives in the cognitive domain is claimed, the cognitive structure itself is not well defined. The taxonomy does not explicitly address the cognitive processes of perception, memory, thinking, and intuition. Further, Bloom's taxonomy is somewhat redundant, for instance, the 'extrapolation' category at the level of understanding in a way replicates the category of 'application of concepts' at the level of application. Another weak point of Bloom's taxonomy is a vague distinction between the levels of application, analysis, and synthesis in terms of their cognitive hierarchy and cognitive weight. In this regard, G. Madaus and his colleagues (1973) proposed to modify Bloom's taxonomy branching out application, analysis and synthesis levels. This ensures a certain degree of flexibility and eliminates a contradiction in the link "application — analysis — synthesis." Some critics of Bloom, for example R. Van Horn (2007), indicate that the taxonomy is quite general in nature and does not reflect the characteristics of specific methodological aspects, for instance, the development of students' problem solving skills. In this regard, Horn proposed a problem solving algorithm that was aligned with Bloom's taxonomy.

Another modification was proposed by W. Gerlach and A. Sullivan (1967). If Bloom's taxonomy is mainly based on the hierarchy of internal mental actions (e.g., understanding, analysis, synthesis), Gerlach-Sullivan's taxonomy is built upon the concept of learning behaviors or external procedures that students perform during the learning activities. Based on this assumption, they proposed the taxonomy of learning objectives consisting of the following levels: identification and classification of objects (e.g., an object is an element of a certain set); naming (e.g., a figure is called an isosceles triangle); describing (e.g., reproducing/retelling basic properties of a parallelogram); designing (e.g., constructing a perpendicular line to another line from a given point on a plane); ordering (e.g., sequencing arithmetic operations in a complex numerical expression); demonstrating (e.g., using external and internal representations of a concept).

Unlike the above taxonomies, the hierarchy of objectives proposed by A. De Block (1975) is a practice-based spatial representation of learning objectives including the following dimensions: learning categories (such as fact, concept, correspondence, structure, method, relationship); connections (e.g., inter- and intradisciplinary connections); levels (e.g., knowledge, understanding, application, and generalization). Using three dimensions and corresponding elements, one can make 72 combinations of learning objectives. Teachers appreciate the level of detail in designing learning objectives using De Block's taxonomy such as, for instance, learning concepts at the level of understanding through interdisciplinary connections.

Another well-known taxonomy of objectives in the development of intellectual abilities of students is the model proposed by J. Guilford (1967). He used a spatial model based on the following dimensions: content, products, and operations. The content component includes various types of knowledge representation: figurative, symbolic, semantic, and behavioral. Figurative content includes visual-practical, audio-visual, and physical representation of knowledge. Symbolic content is based on signs and abstract representation. Semantic content consists of verbal (both written and oral) knowledge representation and behavioral content, includes emotions, communication, and interaction.

The products of intellectual activity are presented at the following levels: units, classes, relations, systems, transformations, and implications. Units include elements, objects, and/or parts of a whole. Classes represent the set of elements that are grouped according to certain criteria. Relations consist of connections between various elements, sets, and classes. Systems represent integrative holistic structures. Transformations are methods of changing elements and sets. Implications include investigation, conclusions, predictions, etc.

The operational component of the model includes the following categories: knowledge, memory, convergent thinking, divergent thinking, and evaluation. The knowledge category represents information retrieval and recognition. Memory includes acquisition and storage of information. Divergent thinking suggests a variety of ideas and methods of problem solving, whereas convergent thinking could be considered as a strictly logical thinking. Evaluation represents assessment of the solution, its verification with the initial conditions, and control. As in De Block's model, every combination of three dimensions in Gilford's taxonomy represents a certain learning objective, for instance, development of students' evaluation skills in understanding semantic units of knowledge.

Gagne and Merrill (Gagne, 1964; Merrill, 1971) proposed a taxonomy that integrated different domains of learning: cognitive, affective and psychomotor. It contained a hierarchy of objectives, which described four different levels of learning behavior: the level of emotional behavior that encompasses affective characteristics (e.g., surprise, joy, inspiration, confusion); the level of psychomotor behavior that represents physical actions of students; the level of memory that involves mainly recalling and recognizing; and the level of complex cognitive behavior that includes a system of learning objectives on classification, analysis and problem solving.

Marzano and Kendall (2006) proposed a taxonomy that consisted of three systems and a knowledge domain. The systems were a self-system, a metacognitive system, and a cognitive system. The self-system includes primarily affective dispositional characteristics such as beliefs about the importance of knowledge and efficacy as well as emotions associated with knowledge. The metacognitive system addresses students' self-monitoring of clarity, accuracy, and execution of knowledge along with the specification of learning goals. The cognitive system consists of knowledge retrieval, comprehension, analysis, and knowledge utilization. The knowledge domain includes three categories: information, mental procedures, and physical procedures.

Despite some flaws, the discussed taxonomies provide tools that make it possible to identify the starting positions in the design of learning objectives. The next step for a teacher-engineer is to take an initiative and design his/her own taxonomy that reflects personal experiences, preferences and teaching style.

The process of analyzing and designing learning objectives leads a teacher-engineer to careful selection and construction of learning tasks and activities. Learning tasks create a link between learning objectives and learning outcomes. Teachers persistently check for students' understanding in order to support student learning (Shepard et al., 2005). They often need to assign a variety of tasks that would assess students understanding. According to Boston and Smith (2009), "different kinds of tasks lead to different types of instruction, which subsequently lead to different opportunities for students' learning" (p. 122). In addition, "tasks convey messages about what mathematics is and what doing mathematics entails" (NCTM, 1991: 24). That is why choosing correct tasks is important for engineering of learning.

According to Smith and Stein (1998), "tasks that ask students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that require students to think conceptually and that stimulate students to make connections lead to a different set of opportunities for student thinking" (p. 269). Quite often, students are given only the tasks that require memorization or procedure that suggests following an algorithm. But students should be asked to fulfill high-level tasks as well. Stein et al. (2000) proposed a cognitive demand framework. They separated the low-level cognitive demands from high-level ones where memorization and procedures without connection fall to the low level, while procedures with connections and doing mathematics represent the high-level of cognitive demand. More specifically, the tasks at the level of memorization involve reproducing previously learned facts, rules, formulae, or definitions. This level may also include committing facts, rules, formulae, or definitions to memory. The tasks at this level "cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure" (Smith & Stein, 1998). Usually, such tasks have no connection to the meaning of facts, rules, formulae, or definitions. Procedures without connection are algorithmic by nature and require limited cognitive demand for completion. Moreover, such tasks do not require connection to the concepts or meaning that underlies the procedure. Procedures with connections focus students' attention on understanding of concepts and ideas. Such tasks are usually represented in multiple ways (e.g., numerical, visual, concrete, symbolic) and require making connections among multiple representations. The highest level of cognitive demand — reasoning (or doing mathematics) — requires non-routine, non-algorithmic thinking to explore and to understand the nature of mathematical concepts, processes, or relationships. Such tasks usually require students to access relevant knowledge in order to solve the problem, to examine constraints that may limit possible solutions. Reasoning tasks demand significant cognitive effort due to the unpredictable nature of the problem solving process at this level.

A modified version of the cognitive demand model presented by Tchoshanov, Lesser and Salazar (2008) includes three levels: (1) facts and procedures; (2) concepts and connections; and (3) models and generalizations. Every level is specified by the list of descriptors presented in the table below (Table 6).

Table 6. Levels of cognitive demand descriptors

Levels of Cognitive Demand	Descriptors
Facts and Procedures	• Recognize basic terminology and notation • Recall facts • State definitions • Name properties and rules • Do computations • Make observations • Conduct measurements • Simplify and evaluate numerical expressions • Solve routine problems
Concepts and Connections	• Select and use appropriate representation • Translate between multiple representations • Transform within the same representation • Transfer knowledge to a new situation • Connect two or more concepts • Explain and justify solutions to problems • Communicate major mathematical ideas • Explain findings and results from the analysis of data • Solve non-routine problems
Models and Generalizations	• Generalize patterns • Formulate mathematical problems • Generate mathematical statements • Derive mathematical formulas • Make predictions and hypothesize • Design mathematical models • Extrapolate findings from data analysis • Test conjectures • Prove statements and theorems

In order to illustrate the process of task selection and design at different levels of cognitive demand, let us consider the following case focused on fraction division. How much thinking is required to solve the task below?

Task 1. What is the rule for fraction division?

A. $\frac{a}{b} \div \frac{c}{d} = \frac{ac}{bd}$ B. $\frac{a}{b} \div \frac{c}{d} = \frac{ab}{cd}$ C. $\frac{a}{b} \div \frac{c}{d} = \frac{cd}{ab}$ D. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

One would say "a little" or "no thinking is required" to solve this item. It demands only *memorization* of the fraction division rule. All middle school teachers who were given this task responded correctly (choice D). Task 2 below addresses the same mathematical procedure — fraction division.

Task 2. Which of the problems below represents the operation $1\frac{3}{4} \div \frac{1}{2} = ?$

A. Juan has a piece of rope $1\frac{3}{4}$ feet long and cuts it in half. At what length should he cut the rope?

B. Maria has $1\frac{3}{4}$ liters of juice. How many $\frac{1}{2}$ liter containers can she fill?

C. A boat in a river moves $1\frac{3}{4}$ miles in 2 hours. What is the boat speed?

D. Daniel divides $1\frac{3}{4}$ pounds of coffee evenly between 2 customers.

How many pounds of coffee will each customer get?

What is the difference between Task 2 and Task 1? How much and what kind of thinking is required to solve Task 2? Obviously, Task 2 is more cognitively demanding: it requires *understanding* of the fraction division concept. 72% of the same sample of middle school teachers was able to solve this task correctly (Choice B). Task 3 below deals with the same fraction division procedure.

Task 3. Some students mistakenly divide two fractions in the following way: $\frac{a}{b} \div \frac{c}{d} = \frac{ac}{bd}$. If *a, b, c, and d* are positive integers, choose the correct answer:

A. This equation is always true.
B. This equation is true when *c=d.*
C. This equation is never true.
D. This equation is true when *ad=bc*.

This task is different from the Tasks 2 and 3 because it requires thinking at a higher cognitive level — *generalization*. Only 41% of the same sample of middle school teachers responded correctly (choice B). Not surprisingly, the majority of incorrect responses fell under choice C. Teachers' low performance on Tasks 2 and 3, in particular, showed that they lack understanding of very basic and fundamental idea of school mathematics — fraction division.

Understanding the design of learning objectives and its close connection to the selection and construction of learning tasks and didactical situations is a critical step in the engineering of learning. Traditionally, tasks are selected among those available in textbooks. Students' engagement in problem solving is primarily considered as finding solution to a given task using prescribed methods (e.g., rules, formulas, and conventional algorithms.

The *holistic problem solving approach* (Tchoshanov, 1996) requires students to be involved in the complete problem solving cycle that includes the following three major stages:

1) posing a problem;
2) solving a problem; and
3) debugging a solution.

Implementation of all three stages constitutes a holistic problem solving, as opposed to the traditional fragmented problem solving approach. The first and the most important stage is students' engagement in posing the problem to be solved. Problem posing lies at the heart of learning (Polya, 1945; Silver, 1994). If a student is involved in designing of a task/problem, s/he will be highly motivated to solve it. In traditional teaching, major emphasis is placed on the second stage — solving a problem; stages of posing a problem and debugging a solution are usually done by a teacher or overlooked and ignored. This significantly reduces the efficiency of problem solving as a learning activity. Each stage in the structure of a holistic approach to problem solving can be broken down into steps. Problem posing refers to both generation of new problems and reformulation of given problems (Silver, 1994). The process includes the following steps: analyzing the situation from which a problem may arise; defining a problem or question; identifying constraints; collecting data if needed; formulating a problem; reviewing a problem (Tchoshanov, 1996). The second stage is similar to the well-known problem solving plan established by Polya (1945): understanding the problem; devising a plan; carrying out the plan; looking back. The debugging stage includes the following steps: recognizing an error in the solution; localizing the error; identifying a cause of the error; devising a plan to debug the error; implementing the plan and checking the fixed solution (Tchoshanov, 1996).

Another important element related to designing learning tasks is constructing didactical situations. Building on the work of Makhmoutov (1975), Okon (1990), Brousseau (1997), and other scholars, the didactical situation is defined as a purposefully designed fragment of teaching that aims at engaging students in a learning task, problem, and/ or activity. There are different classifications of didactical situations proposed by different scholars. Didactical situations could be categorized as theoretical and/or practical; inductive and/or deductive; routine and problem-based, etc.

We propose the classification of didactical situations based on the idea of cognitive conflict/tension (Tchoshanov, 1996). We consider cognitive conflicts between the four main constructs: (1) known (K — prior knowledge); (2) unknown (U — new knowledge); (3) theoretical knowledge (T); and (4) applied/ practical knowledge (P). Thus, we distinguish the following main types of cognitive conflicts and corresponding didactical situations as depicted in Figure 12:

DS-1: KT-KP. Between the theoretical known (KT) and the practical known (KP);

DS-2: UT-UP. Between the theoretical unknown (UT) and the practical unknown (UP);

DS-3: UP-UT. Between the practical unknown (UP) and the theoretical unknown (UT);

DS-4: KT-UP. Between the theoretical known (KT) and the practical unknown (UP);

DS-5: KT-UT. Between the theoretical known (KT) and the theoretical unknown (UT);

DS-6: KP-UP. Between the practical known (KP) and the practical unknown (UP);

DS-7: KP-UT. Between the practical known (KP) and the theoretical unknown (UT);

DS-8: KP-KT. Between the practical known (KP) and the theoretical known (KT).

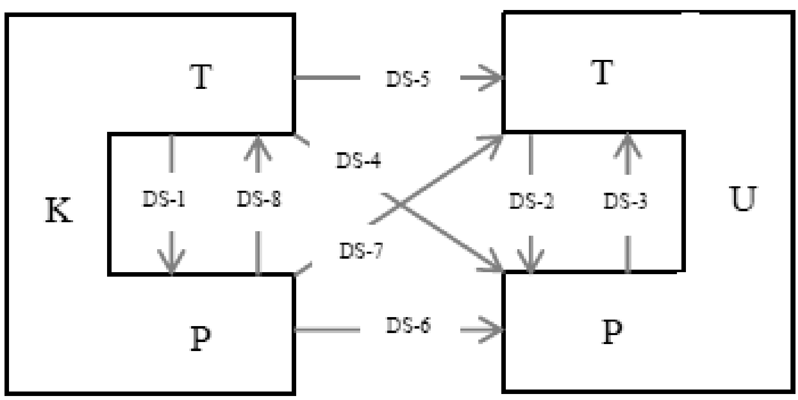

Fig. 12. Classification of didactical situations (DS) based on cognitive conflicts between known (K) and unknown (U) knowledge (P — practical and T — theoretical)

Let us illustrate the selected types of didactical situations (DS) using the following task: "Let's start with the following true statement: *10 + 6 — 16 = 15 + 9 — 24*. Then let's factor 2 out in the left side and factor 3 out in the right side of the equation: *2 * (5 + 3 — 8) = 3 * (5 + 3 — 8).* Let's then reduce both sides by the same expression *(5 + 3 — 8).* We obtained *2 = 3*. Do you agree with this line of reasoning? Explain why you agree or disagree."

DS-1: A student has learned the theoretical rule of the "division by zero is undefined" some time ago. However, s/he is experiencing a cognitive conflict of applying the rule to a familiar practical situation: s/he knows most of the operations (e.g., adding, subtracting, multiplying, factoring, and reducing).

DS-2: A student has just learned a new rule "division by zero is undefined" and s/he is facing a new practical situation where s/he needs to apply division as reduction.

DS-3: A student recognizes that there is something wrong in the final statement "*2 = 3*" but s/he does not know the rule "division by zero is undefined" yet to justify her/his "empirical hypothesis".

DS-4: A student has learned the property of multiplying by zero (*a*0 = 0*) some time ago but s/he has a cognitive conflict of applying this knowledge to a new practical situation.

DS-5: A student knows that multiplication by zero is allowed (prior theoretical knowledge) and assumes the same for division — why not!? The result "*2 = 3*" confronts her/his assumption and leads to a new theoretical knowledge — "you cannot divide by zero".

DS-6: This type of didactical situation is similar to DS-4 type with one significant difference: while studying the topic "Multiplication" a student applied the rule *a*0 = 0* in familiar practical situations. While learning a topic on "Division" s/he faced a new situation where the previously learned multiplication rule occurs in a new "veiled" form *a*0 = b*0.*

DS-7: A student knows how to solve the problem "*2=3*" by using the "multiplication by zero" property: if *2*(5 + 3 — 8) = 3*(5 + 3 — 8),* then *2*0 = 3*0*, which leads to a true statement *0 = 0*. Now the student is asked to find another method of solving the problem which might lead her/him to discover the new "division by zero is undefined" rule.

DS-8: A student knows how to solve the problem "*2 = 3*". Now s/he is challenged by another non-routine problem, which requires applying the same "multiplication by zero" or "division by zero" rules.

The practical implication of this approach is to examine possible types of cognitive conflicts between the known and the unknown, the theoretical and the practical knowledge in order to provide a systematic approach to classification and design of didactical situations.

3.2. Cognitive Tutoring, Representations and New Literacies

Technology skills have become increasingly important for active participation in different social spheres. Considering the need to prepare students to become fully functional citizens, school curricula have integrated technology innovations meant to equip students for the work while also developing critical thinking. Cognitive tutoring is a good example of technology innovation that serves the purpose of preparing students to meet the requirements of modern society.

Derived from the theory of human learning and performance called *Adaptive Control of Thought–Rational* (ACT–R; Anderson & Lebiere, 1998), computer tutoring proved to be an effective method for providing individualized tutoring, incorporating advances in learning sciences into the classroom, testing associated learning principles, and adapting them to the needs of students and teachers. Koedinger and

Cobbert (2006) reported that the students who had used Cognitive Tutor Algebra system performed 15-25% better on the standardized test items taken from the SAT than the control group. Moreover, their findings revealed that 50 to 100% of the students utilizing the cognitive tutoring program performed better on problem solving and representation use. They explained why the Cognitive Tutor Algebra system enhances student motivation: (1) authentic problem situations make mathematics more relevant, interesting, and sensible; (2) the majority of students prefer doing instead of listening and the structure of the Cognitive Tutoring Algebra problems is similar to playing a video game; (3) students feel safer, less threatened, and experience less frustration when the feedback and opportunities for learning are provided by a computer tutor instead of a human tutor; and (4) students feel empowered knowing that they are mastering mathematics. Stemming from the ACT-R theory, cognitive tutors resemble good human tutors, as they are able to monitor individual student performance and learning (Koedinger & Corbett, 2006). In this regard, classroom learning experiences could be greatly enhanced by cognitive tutors, a measure that would be considerably less costly than hiring human tutors for one-to-one instruction.

Koedinger and Corbett further explored the pedagogical benefits of cognitive tutors, which are computer programs based on "cognitive models that represent learner thinking or *cognition* in the domain of interest" (p. 62). The cognitive model relies on a "if …, then" system to represent the strategies that students may apply to solve problems. In identifying the strategy selected by a student, and detecting typical student misconceptions, cognitive tutors are able to provide context-specific instruction and scaffolding that includes immediate feedback. The "if …, then" rule is one of the models used in the computer science discipline known as *knowledge engineering* (Feigenbaum & McCorduck, 1983) along with logical models, frames, and semantic webs to solve complex problems requiring high level of human expertise.

A cognitive tutor utilizes two main features: *model tracing* and *knowledge tracing*. Model tracing involves identifying student`s approach to a problem to provide individualized assistance as immediate feedback. Since just-in-time feedback is crucial for students, the authors corroborate this point by highlighting the studies that found immediate feedback to contribute to accelerated learning with simultaneous increase in student motivation (Corbett & Anderson, 2001; Schofield, 1995). Unlike model tracing, knowledge tracing monitors each student`s learning applied to problem-solving and decides whether s/he is ready to start working on a higher-level task. If the student is not ready yet, the computer program will provide more practice before moving on to the next level. As shown by Huber (1990) and Corbett et al. (2001), knowledge tracing is an important feature of cognitive tutors because it offers students the opportunity to learn at their own pace and also learn from their own errors without frustrating experiences of stigmatization among peers. Even though cognitive tutors have been found to be very effective, experienced teachers still play a key role in the classroom enhanced by cognitive tutors because they facilitate the learning process by helping students to make connections between computer-based activities and other types of classroom activities designed to promote interaction among students. From a constructionist perspective, the ability to make meaningful connections between what is being learned and the social context is critical in the process of knowledge appropriation (Kafai, 2006). Through interactive classroom activities experienced educators strengthen the implementation of cognitive tutors in assessing students' prior knowledge, debugging misconceptions, helping students to recognize the applicability of the content learned to real life situations, and also fostering students' self-monitoring and reflective thinking processes (metacognition). Cognitive tutors with teacher guidance can greatly contribute to the assimilation, accommodation and appropriation of knowledge using graphical, tabular, and symbolic representations (Figure 13: http://www.carnegielearning.com/galleries/4/).

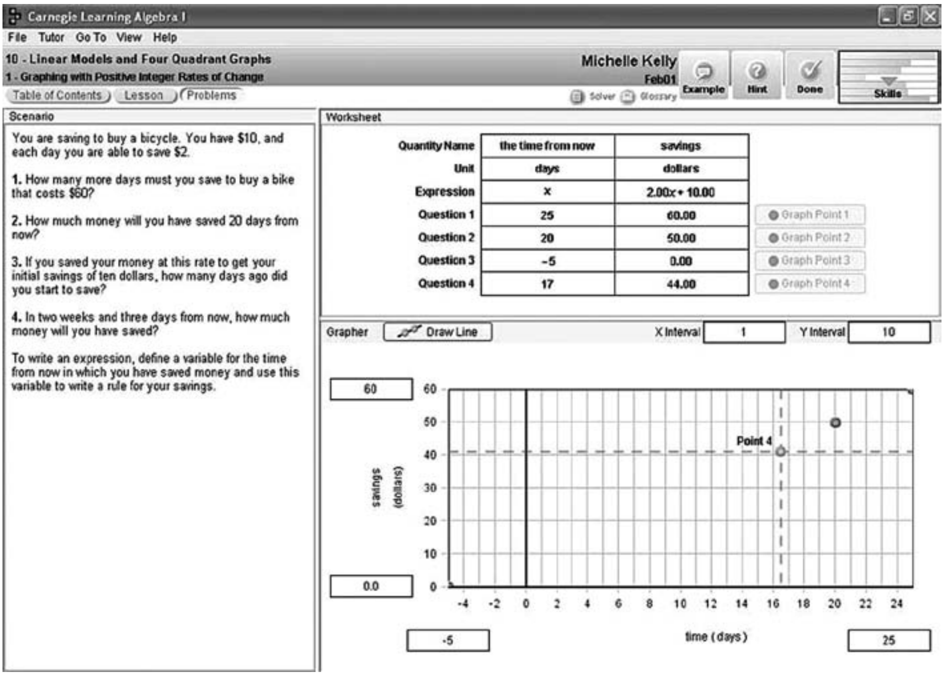

Fig. 13. Screenshot of the Cognitive Tutor Algebra system

The multiple modalities used in the Cognitive Tutor Algebra system (e.g., symbolic, tabular, graphic) are considered to be external representations. However, scholars claim that *representation*[3] could refer to both internal and external manifestations of concepts (Pape & Tchoshanov, 2001). Based on this approach, representations may be thought of as *external* stimuli (numerals, equations, graphs, tables, diagrams, etc.) of concepts or *internal* cognitive schemata — abstractions of ideas that are developed by a learner through experience. Representation could also refer to the act of externalizing an internal, mental abstraction. The key question is the relationship between external and internal representations in learning: how

[3] The section on representations is adapted from author's collaborative work with S. Pape (Pape and Tchoshanov, 2001)

students' internal schemata assimilates external representations, and how new external representations help students to accommodate their emerging internal representations. Figure 14 depicts the interplay between students' internal and external representations in learning a basic concept of *five,* as an example.

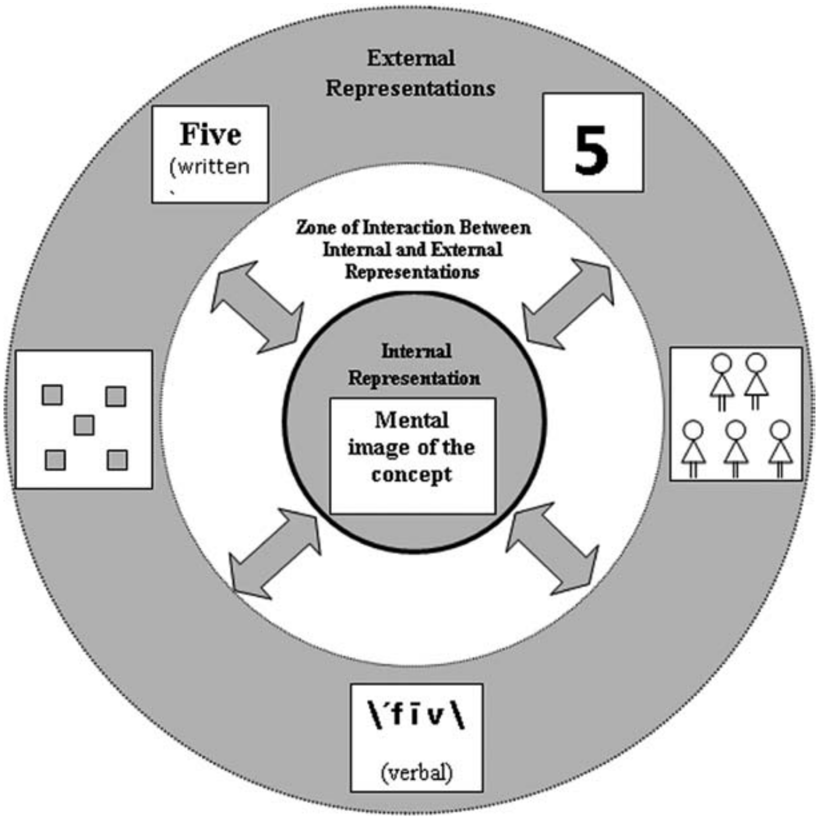

Fig. 14. The relationship between external and internal representations in developing understanding of the concept *five* (modified from Pape and Tchoshanov, 2001)

Both local and national standards require that students should be able to create and use various forms of representations flexibly to organize, record, and communicate ideas; to select, apply, and translate among representations; to solve problems, to investigate, model, and interpret real-world phenomena. The pathway toward the flexible use of multiple representations, however, is challenging. The development of students' representational thinking is a two-sided process, an interaction of internalization of external representations and externalization of mental images (Figure 14). There is a mutual influence between the two forms of representations: the nature of the external representation influences the nature of the internal one, and vice versa.

Symbol systems support the cognitive activity by reducing the cognitive load, clarifying the problem space, and revealing immediate implications. Thus, symbols or symbol systems help an individual to solve a problem, or to provide an explanation, prediction, or justification (Perkins & Unger, 1994: 6). At the same time, simplistic external representations could engender simplistic understandings, while complex external representations may facilitate students' understanding of more complex phenomena, and vice versa. Finally, representation is an inherent social activity. When students are asked to represent data in a graph, the graph should not be a static end result, but rather a vehicle for further discussion to help them establish a justification within a social context. Therefore, *representational thinking* is learner's ability to construct, interpret, and communicate effectively with both forms of representations, external and internal, individually and within social situations (Pape & Tchoshanov, 2001).

There are various views on the relationship between external and internal representations. On the one hand, the advocates of a "picture" theory of representation (Mitchell, 1994; Wileman, 1980) argue that there is no difference between external and internal (mental) representations: mental representation is equivalent to what it represents. On the other hand, some researchers (Arnheim, 1969; Mc Kim, 1972) believe that the development of students' thinking is directly connected to their ability to operate with mental images (e.g., seeing, imagining and idea-sketching).

Cobb, Yackel, and Wood (1992) criticize the "picture" theory and claim that this representational view begins with experts' ideas and conceptions and attempts to reproduce these ideas within instructional materials. Therefore, when learning a procedure using manipulative materials, for example, learner's task is to create a mapping between the manipulation of these concrete materials and the internal abstraction. From a constructivist perspective, the mapping between the concrete materials and the algorithm requires intensive social co-construction of meanings. Teachers and students co-construct their understanding of the steps in establishing the mapping while manipulating with the materials.

The conceptualization of representation is also based on recent findings in the theory of cognition and brain research (Caine & Caine, 1994; Chabris & Kosslyn, 1998). According to these studies, the brain works more effectively while making representational patterns for encoding (internalizing) and decoding (externalizing) information. For example, it is almost impossible to memorize a multi-digit number 1123581321345589, unless the students recognize and follow the Fibonacci pattern, where each succeeding term is the sum of the two preceding. "Seeing" this relationship means that the students can easily internalize (e.g., understand) and externalize (e.g., reproduce) the number based on the pattern.

Unfortunately, as opposed to the varied and complex patterns generated in the human brain, most of the content offered to students is typically presented in abstract/ symbolic and linear forms. The cognitive capacity of the human brain, however, more closely resembles multiple representational patterning: combinations of concrete, visual, and abstract. It seems reasonable that the language of the brain consists of multiple representations. Therefore, the development of students' thinking requires a multiple representational approach.

As discussed before, the critics of the "representational view of mind" (Cobb et al., 1992) believe that it is problematic because the instructional materials developed by experts embody their conceptions of mathematical ideas, and may not be readily available or understandable to the novice. Only when the use of representation(s) is built up in the classroom as a cultural activity students are able to come to an understanding of the meanings of the concrete materials and the associated symbolism. That is, for external manifestations and the internal concept they represent to be connected with student's experiences, representations must be viewed as vehicles for exploration within social contexts that allow for multiple understandings of content (Seeger, 1998).

Students need to practice the use of multiple representations in various situations. Practicing representation(s) must be a part of the social environment; "… learning to construct and interpret representations involves learning to participate in the complex practices of communication and reasoning in which the representations are used" (DiSessa et al., 1991; Greeno & Hall, 1997: 361). Initial students' attempts to portray phenomena using representations often involve non-standard symbolism that is negotiated and refined through the discourse with peers and teacher (DiSessa et al., 1991).

Representations must be thought of as tools for cognitive activity rather than products or the end result of a task. For example, the models (e.g., graphs or other pictorial representations) produced may be used to help students to explain or justify an argument. "When representations are used as tools for understanding and communication, they are constructed and adapted for the purposes at hand" (Greeno & Hall,1997: 362). Representations allow individuals to track intermediate results, ideas, and inferences. Since an external representation embodies the important relationships presented in data or a word problem, they lighten the cognitive load of the individual and serve to organize individual's further work on a problem. Given the representation, the learner may work on the alternative parts of the problem. Representations then may be used to facilitate an argument and to support conclusions.

Finally, the sequence and combination of multiple representations are important. A study on trigonometric problem solving and proof by high school students showed an impact of representational sequence on students' understanding (Tchoshanov, 1996). The first group of students ("pure-analytic") was taught by a traditional analytic (algebraic) approach to solving and proof. The second group ("pure-visual") was taught using a visual (geometric) approach with enactive (i.e., geoboard as a manipulative aid) and iconic (pictorial) representations. The third group ("representational") was taught by a combination of analytic and visual means using translations among different representational modes. The representational group scored 26% higher than the visual group and 43% higher than the analytic group. This experiment also showed that the students in the "pure" (analytic and/or visual) groups "stuck" to one particular mode of representation; they were reluctant to use different representations. For instance, the students in the pure-visual group tried to avoid any analytic solutions: they were "comfortable" only if they could use visual (geometric) problem-solving and proof techniques.

The students in the representational group were more flexible “switching” from one mode of representation to another and discussing solutions with their peers for better understanding of mathematical concepts. Therefore, any intensive use of only one particular mode of representation does not improve students’ conceptual understanding and representational thinking. This study also proved the importance of students’ social interaction using different models (e.g., concrete, visual, and abstract) in the process of developing representational thinking.

The development of students’ understanding and representational thinking requires the combination of multiple representations as well as the interaction between internal and external representations. Through activity the learner begins to abstract meaning. However, we must be cautious not to advocate the position that this abstraction occurs solely within the individual. It is through the externalization of these abstractions within social environments that learners begin to negotiate the meanings of their understandings and refine these representations accordingly. The interrelated processes of internalization and externalization are called *cognitive representation* which reflects both the process (internalization) and the product (externalization) of representational thinking.

In the digital age, the idea of cognitive representation is closely connected to the emerging concept of new literacies. Technology acts as a platform for new literacies and shared learning. As a result of the expanding technological repertoire, the ‘identity’ of literacy is also continuously expanding. Learning sciences position literacy within the new and continuously advancing technological landscape. As a result of the new intersection created by the merging of literacy and technology, the new problem space was created (Reinking et al., 1998). New opportunities for communication are provided through electronic mail, discussion threads, interactive chats and collaborative data bases. New literacy and technology discourses have emerged as students engage in individual and collective enterprise (Palincsar & Ladewski, 2006). Nixon (2003) recognizes the expanding definitions of “literacy as a repertoire of practice for communicating and accomplishing goals in particular social and cultural contexts” (p. 300). Palincsar & Ladewski (2006) discuss multiple forms of literacy that are needed for students to access, interpret, decode and manipulate various technological tools. Expanding on the idea of new literacies, Perez Tornero and Varis (2010) analyze an impact of digital age and information society on *media literacy* which “is required by the convergence of media, both analogical and digital, and new multimedia platforms” (p. 33).

New literacies require sets of skill and strategies for successful use and adaption of the rapidly changing information and communication technologies. The changes are guided by ten principles that include global community, internet use, literacy and technology transactions, centrality of strategic knowledge, social constructs of learning, and the role of a teacher (Leu et al., 2000). While the role of a teacher is listed lastly, this component is certainly not the least, as teachers are central to facilitating students’ navigation through the new literacies that link education to the expanding technological landscape. It is imperative that teachers realize and accept their important role in facilitating students’ operational, cultural and critical literacy in the digital age.

Operational, cultural, and critical literacies are three dimensions of literacy that are related to learning sciences (Palincsar & Ladewski, 2006). *Operational literacy* consists of one's competences with tools, procedures and techniques in written language proficiency, particularly in navigating hypermedia text which is primarily reader-driven. When students are competent with these features, they are better equipped to be productively engaged in the use of new technologies to develop deeper understanding of a content domain. Therefore, studying literacy in today's technological world necessitate a consideration for incorporating multimedia, hypermedia and hypertext environments which are "fluid, spatial, decentered, bottom up and playful" (Ibid.: 302), as well as being "reader driven," giving the readers the freedom to choose the ways to navigate through such environments. These features, in turn, suggest that new literacies are constantly changing as new technologies emerge.

Cultural literacy is defined as a "competence with the meaning system of a practice" (Palincsar & Ladewski, 2006: 302) and reflects the key idea that "literacy is not a unitary construct but rather is embedded in and develops out of the social practices of a culture" expressed by Gee (1991, 1996). Based on this perspective, the study of literacy opens in multiple ways including global media culture that became an integral part of the youth's affiliations and identities. Hull's (2003) ethnographic project "Digital Underground Storytelling for Youth" is an example of how the cultural literacy could be reinforced in the community. The aim of this project was to "close the digital divide and provide youth with access to new technologies and a context in which they can create, envision, and revise, represent themselves and their ideas, and learn the power of communication" (Palincsar & Ladewski, 2006: 308). Moje et al. (2004) investigation addresses the importance of links between the worlds of family and school and illustrates the power of bridging perspective to the study of learning. According to this perspective, individuals occupy several spaces, each of which offers an opportunity to engage in literacy learning. Home, community, and peer networks constitute the *first space,* whereas the *second space* refers to the contexts experienced in schools, workplaces, and churches, where more formalized language and literacy are used. The bridging context between these two spaces is called the *third space*. The cultural literacy capitalizes on the *funds of knowledge* defined as an intellectual and social knowledge existing in families and communities and "ways of using language and print literacy that shape students interactions with texts in and out of school" (Palincsar and Ladewski, 2006: 399). Although the students' funds of knowledge are rich with possibilities to build connections between in- and out-of-school contexts and deepen understanding of content knowledge, teachers seldom utilize these funds in classroom. Teachers need to be aware and "develop a third space by engaging students in experiments, discussions, and reading and writing activities that focus on texts and experiences of different communities" (Ibid.: 399).

Critical literacy. The ability to attend to how texts represent the self and others is a critical literacy. Information and communication technologies have opened new vistas for research, inclusive of exploring learning from media and texts, having implications for effective engineering of learning and enhancing students' proficiency.

New technologies have shifted learning spaces from linear to hypermedia learning; from instruction to construction and discovery; from teacher-centered to learner-centered classrooms; from focusing on students' ability to absorb material to learning how to navigate and how to learn; from schools to lifelong learning; from one-size-fits-all teaching to customized, differentiated learning; from learning as a torture to learning as fun; from teachers as transmitters to teachers as facilitators (Tapscott, 2009). Teachers continue to be the critical component of students' literacy acquisition and development along with new technologies that are rapidly and increasingly gaining influence in today's classrooms.

Understanding the media and technology influence on literacy learning is critical to teaching. Students embrace technology and often are more comfortable with its use than teachers are. Teachers need to understand the emerging digital reality and use it to their instructional advantage. Bridging digital culture to the classroom context impacts instruction in positive ways by increasing student engagement in learning. The new generation of children is the first to grow up completely surrounded by technology (Tapscott, 2009) and teachers should learn to address and take advantage of the new literacies and the learning spaces that the digital era has provided. To do so, teachers need to advance their own levels of operational, critical, and cultural literacy. Therefore, providing professional development and training is central to support teachers in accepting and acting in their important roles as facilitators of the new literacies.

New literacies also advance a reflection upon Lave and Wenger's (1991) situated learning theory, which places the importance on the authentic social interaction as being fundamental to learning. The authentic social interactions for literacy learning are taking place through the use of new technologies, creating a community of practice, which reinforces "developmentally appropriate versions of the situated and meaningful practices of experts" (Sawyer, 2006: 5). When students are engaged in authentic and situated practices to solve problems through the use of new technologies, they socialize with communities that share similar interests resulting in valuable benefits from others who might be more knowledgeable in that particular discourse. Authentic experiences create a learning culture that supports students' learning and thinking.

Authentic practices have gained increasing attention from educators and researchers. It is believed that the engagement in authentic practice activities is beneficial for students for three main reasons: they bridge disciplinary knowledge to the outside world; they increase students` motivation by making content meaningful; they enable students to understand the structure of knowledge (e.g., epistemology). Edelson and Reiser (2006) discuss the importance of engaging students in authentic practices that mirror the practices of scientists. An authentic practice is described as "the activities through which experts in a domain apply their understanding to achieve valued goals... when we talk about engaging students in authentic practices, we are talking about developmentally appropriate versions of the authentic practices of experts" (Edelson & Reiser, 2006: 352). However, implementation of authentic practices suggests pedagogical and practical challenges. Among the pedagogical challenges one can mention: first, teachers may have never incorporated these practices and second, teachers have limited time and resources to support the implementation. To respond to these challenges, a systemic perspective in design is needed to include the following strategies: (a) situate authentic practices in meaningful contexts; (b) reduce the complexity of authentic practices or cognitive load; (c) make implicit elements of authentic practices explicit; and (d) sequence learning activities according to developmental progression. Along with the strategies, three key elements in design research are considered critical in the implementation of authentic practices: *classroom activities* (curriculum); *tools and resources*; and *social structures,* which relate to communication, interaction, and learning culture. Most importantly, authentic practices require authentic assessment which will be addressed later in Chapter 3.4.

3.3. Research-Based Strategies in Engineering of Learning

Engineering of learning depends on many factors including but not limited to the knowledge of learning theories and learning sciences that will inform outcome-oriented design of learning objectives, engineering of content, and assessment toward creating effective learning environment. Along with the guiding principles of learning (discussed in Chapter 2), learning theories and learning sciences inform a teacher-engineer about research-based strategies to support learning.

Below we consider some research-based strategies to address the guiding principles of learning in engaging students' prior knowledge, connecting factual knowledge and conceptual understanding, and fostering students' meta-cognitive and self-monitoring abilities.

Strategies to engage students' prior knowledge

In order to build on students' prior knowledge and experiences, a teacher-engineer should design and construct teaching products and select instructional materials according to strategies to ensure:

- right level of difficulty
- signaling
- varying content and complexity
- contiguity
- minimizing cognitive load.

Let us consider the strategy which suggests the use of learning materials at the right level of difficulty. The 'right level of difficulty' means that the learning material should not be too easy or too complex. If the learning material is too easy, a student is not challenged enough. If the material is too complex, a student may give up. In both cases, student motivation, attention, and engagement will be significantly decreased (Ambrose et al., 2010; Metcalfe & Kornell, 2005; Wolfe et al., 1998). The learning material should be at a level of the student's zone of proximal development (Vygotsky, 1978), so that s/he could learn and understand new material with some support and scaffolding. The same strategy should be applied while designing assignments and assessments. Assignments should not be too difficult or too easy. The 'right level of difficulty' in case of assignments and assessments means that students cannot complete the assignment effortlessly. However, they can successfully complete it with some cognitive effort, support and/or scaffolding. If assignments/assessments are too difficult or too easy, students may get frustrated or bored (Ambrose et al., 2010; Metcalfe & Kornell, 2005; Wolfe et al., 1998).

Along with the right level of difficulty, before starting a lesson a teacher-engineer should provide an overall structure and highlight the organization of the lesson. This strategy is called 'signaling' and includes using outlines, section headings, bullets, which draw students' attention to the most important points in the lesson (Harp & Mayer, 1998; Mautone & Mayer, 2001; Mayer, 2005). Moreover, the learning material should be presented in a way that the points that require attention are highlighted, trying to avoid irrelevant information (even if it might be artistically and aesthetically appealing). Appealing but irrelevant information (e.g., text and graphics) distracts students' attention and leads to missing important points (Kalyuga, et al., 1999).

Opportunity to work on problems that vary in content and complexity will help students to develop multiple layers of knowledge including facts, procedures, concepts, and models, and to connect these layers (Rouet, 2006; Spiro et al., 1991). Moreover, a teacher-engineer should design a learning environment where students could work collectively on challenging real-world problems. In a cooperative problem-solving activity, student's prior knowledge should be linked to challenging real-world problems, which will motivate student and facilitate learning by applying multiple levels of knowledge and skills (Johnson & Johnson, 1999, 2009; Karau & Williams, 1993; Hodara, 2011).

The contiguity strategy suggests introducing closely in time and space the concepts and ideas that need to be connected. By implementing this strategy, a teacher-engineer will make associations stronger, for instance, when corresponding words and images are presented simultaneously rather than successively (Mayer, 2005).

The 'minimizing cognitive load' strategy recommends to divide complex learning material into smaller parts, thus students learn better. This strategy is increasingly important in designing materials for flipped instruction and other multimedia learning environments. While designing narrated screencasting or animation, a teacher-engineer should present it in segments rather than a single continuous unit, so that students could control it at their individual pace; this will help to avoid overwhelming students with too much information at once (Mayer, 2005; Mayer & Moreno, 2003).

Strategies to develop students' procedural fluency within the conceptual framework

The strategies to connect students' factual knowledge and conceptual understanding include but are not limited to:

- desirable difficulty
- cognitive conflict
- adaptive fading
- in-depth questioning
- multiple representations
- engaging in reading and writing
- generation strategy
- timely constructive feedback.

The 'desirable difficulty' strategy requires effortful cognitive processing by students in learning new knowledge. The learning material at the desirable level of difficulty will make it more memorable. Therefore, rather than introducing the learning material in the same order as it is in a textbook, a teacher-engineer should modify the material presentation to facilitate students' active information processing. Moreover, learning is enhanced when students put additional effort to organize the material themselves, which promotes long-term memorizing of information (Bereiter & Scardamalia, 1985; Bjork, 1988).

The 'cognitive conflict' strategy suggests that in-depth learning is often achieved by engaging students in problem solving situations that are non-routine, paradoxical, and/or counterintuitive to their current knowledge. When students encounter situations that are in dissonance with their existing schemata, a cognitive conflict occurs that could lead to a conceptual change in student's learning and understanding. A teacher-engineer should design situations of cognitive conflict by presenting paradoxes, refutations, and/or asking students to predict an answer, knowing that students' responses would be most likely conflicting with the solution (Chinn & Brewer, 1993,1998; Eryilmaz, 2002; Guzzetti, 2000; Hynd, 2001).

A teacher-engineer should alternate examples (that illustrate a solution) and problems (that students have to solve on their own). Illustrative examples are helpful for low-achieving students. Research shows that fading (or gradual elimination) of examples depending on student performance (adaptive fading) leads to better knowledge retention, compared to fading of examples in the same manner for all students (fixed fading) (Kalyuga et al., 2001; Salden et al., 2009; Schworm & Renkl, 2002; Trafton & Reiser, 1993).

Another research-based strategy in promoting student learning and understanding is an in-depth explanatory questioning technique. In-depth questions include cause-and-effect questions, 'why or why-not' questions, 'what-if' questions, etc. While using the in-depth questioning technique, a teacher-engineer should encourage students to 'think aloud' by speaking and/or writing their explanations to answer the questions (Craig et al., 2006; Graesser & Person, 1994; Pressley et al., 1992).

The use of multiple representations (including concrete, abstract, graphical, descriptive) is an important strategy in building students' conceptual understanding. Most of low-achieving students may understand a concept with concrete examples using manipulatives. However, using only concrete representation will limit student learning. A teacher-engineer should gradually switch concrete examples into abstract representations (e.g., symbols, formulas, equations) to help students transfer knowledge to new situations (Goldstone & Son, 2005; Kaminiski, Sloutsky, & Heckler, 2006; Richland, Zur, & Holyoak, 2007). At the same time, a teacher-engineer should connect graphical representations (e.g., graphs, pictures, videos) with descriptive representations of a concept (rather than simply presenting the text alone) to support student learning. Following the recommendation of the contiguity strategy, graphics and accompanying textual description should be presented close in space and time (Clark & Mayer, 2003; Mayer, 2001; Mayer et al., 2005).

Research suggests that involving students in reading and writing is correlated with the improvement in students' critical thinking, complex reasoning and writing skills. Therefore, while designing a course, a teacher-engineer should include assignments in both intensive reading (more than forty pages per week) and writing (more than twenty pages per course) in the syllabus to increase student performance in critical thinking and writing (Arum & Roska, 2011). Along with reading and writing, it is recommended to use quizzes frequently to re-expose students to key concepts in order to actively recall/generate information. This strategy is based on the generation effect reported by Butler and Roedinger (2007) and others. It is also well documented that learning is enhanced, when students construct responses compared to selecting answers among multiple choices. Congruently, timely feedback provided after each quiz/test contributes to student learning and understanding of the material covered in the test (Butler & Roediger, 2007; Dempster, 1997; Pyc & Rawson, 2010; Roediger & Karpicke, 2006a, 2006b). At the same time, it is recommended that timely feedback with clear learning goals should be provided as a formative assessment with the purpose of improving student learning, as opposed to summative assessment with a focus on evaluation of what students have learned (Ambrose et al., 2010). Timely constructive feedback (compared to delayed summative feedback) is important to student learning and significantly contributes to the improvement of students' performance on exams (Ambrose et al., 2010; Black et al., 2003; Dihoff et al., 2004; Kulik & Kulik, 1988; Wiliam, 2007).

Strategies to foster students' metacognition and self-monitoring

Below we will consider the research-based strategies that support students' metacognitive and self-monitoring activities:

- debugging misconceptions
- active information processing
- constant self-monitoring
- mixed practice
- spacing effect
- goal-directed practice.

The debugging misconceptions strategy (briefly discussed in Chapter 2) helps a teacher-engineer to recognize, address, and correct students' common mistakes. In order to correct students' misconceptions, a teacher-engineer should create a bridge between the prior concept and the new one using meaningful examples and model-based reasoning. They could help students to construct new representations different from their initial intuitive conceptions and make them aware of their own misconceptions. Awareness is the first step in helping students to 'fix' their own misconceptions. Next is developing students' epistemological reasoning (beliefs about the nature of knowledge) in order to facilitate conceptual change for revising their own misconceptions. The research also suggests to engage students in Interactive Conceptual Instruction (ICI), which incorporates ongoing teacher-student dialogue and the use of research-based instruments to provide formative feedback, conceptual terrain of student learning including subject matter knowledge and possible misconceptions (Savinainen & Scott, 2002). Once the students have overcome their misconceptions, the teacher-engineer should engage them in the 'arguing to learn' type of classroom discourses to help strengthen their new concept (Savinainen & Scott, 2002; American Psychological Association, 2011).

An active information processing is another research-based recommendation to foster student metacognition and self-monitoring. Learning techniques such as outlining, connecting, and synthesizing information improve student performance (e.g., long-term retention) compared to rereading materials or using more passive techniques. Along with reorganizing and reviewing the material, students may create their own testing situations such as restating the information in their own words and synthesizing information from multiple sources (e.g., lecture notes, textbooks, web resources) (Bransford, Brown, & Cocking, 2000). The research shows that students learn better when they verbally (rather than by typing) explain the material to themselves using self-generated inferences (Ainsworth & Loizou, 2003; Chi et al., 1989; De Bruin, Rikers, & Schmidt, 2007; Griffin, Wiley, & Thiede, 2008; Roscoe & Chi, 2008).

A teacher-engineer should constantly engage students in a variety of metacognitive activities to monitor and control their own learning, including but not limited to assessing the difficulty of the assigned task, evaluating their own strengths and weaknesses, planning their actions, self-monitoring their performance, and assessing the degree to which the task is complete. In order to be more effective, the results of self-monitoring should be shared with the teacher and the peers (Ambrose et al, 2010; Blerkom & Blerkom, 2004; Brown & Frank, 1990; Chang, 2007; Zimmerman, 2001).

The study by Smith and Vela (2001) claims that when the material is studied in one environment, associations are established between what is studied and contextual factors, preventing the transfer of learning. Contrary, when the same material is studied in multiple environments, its associations with one or a few particular locations dissipates. This, in turn, facilitates students' flexible recall of the material in the new and different environments (ibid). The strategy called 'mixed practice', when the student solves problems related to different topics within the same study session, improves student learning compared to the 'blocked practice' where all problems are taken from the same topic (Rohrer, 2009; Taylor & Rohrer, 2010).

The research conducted by Capeda et al. (2008), Kornell (2009), and Rohrer & Taylor (2006) indicates that students learn better when they spread their study over several shorter practice sessions, rather than concentrate it into one longer session. The practice distributed over time results in better retention of material than cramming (Ibid.). The spacing effect increases, if a student is engaged in the distributed practice that focuses on a specific goal. The goal-directed practice supported by the timely targeted feedback, promotes greater learning gains (Ambrose et al., 2010; Ericsson, Krampe, & Tescher-Romer, 2003; Rothkopf & Billington, 1979). Finally, while designing a course, a teacher-engineer should make a schedule of course quizzes, tests and exams, because students benefit more from repeated testing when they expect exams rather than when exams are unexpected (Szupnar, McDermott, & Roediger, 2007).

The above research-based strategies play an important role in the engineering of learning through designing engaged learning experiences and effective learning environments.

3.4. Assessment of Learning Outcomes

The problem of assessment is one of the key issues that directly affects the effectiveness of learning. The overall student success in learning largely depends on how well the assessment is designed and connected to learning objectives and content.

There are different approaches to designing assessment (Arter,1990; Hart,1994; Herman et al.,1992): outcome-based assessment; standard-based assessment; competency-based assessment; performance-based assessment. The main difference between these approaches is orientation either toward a product (e.g., outcome-based and standard-based assessment) or toward the process of learning (competency-based and performance-based assessment), although all these approaches are important links in the teaching and learning chain: standard — competency — performance — outcome.

The ultimate goal of assessment is to strengthen student's responsibility for the process and outcome of learning. Research suggests that the future projection of traditional assessment should be revised toward authentic assessment changing its dimensions from discrete to continuous; from isolated to interdisciplinary; from focusing on a single measure to more diverse assessment; from primarily quantitative to qualitative and integrated assessment; from prescribed to flexible; from standardized to open; and, last but not least, from assessment to self-assessment (Tchoshanov, 2011). Let us briefly describe each of the approaches.

From discrete to continuous assessment. In the traditional system, assessment is seen as a discrete process: it ends at the stage of the final exam. The main idea of the revised approach is that learning is recognized as a continuous process that does not terminate after the final exam. Moreover, the traditional understanding of the assessment as a measure of the end product should be revisited toward the assessment as a process of achieving the desired outcome. With this revision, for instance, it becomes apparent that students have a right to make mistakes in the process of learning and correct mistakes as they progress toward achieving the desired learning outcomes.

From isolated to interdisciplinary assessment. The traditional assessment is usually aimed at testing students' skills and knowledge of a specific topic within a given subject domain. In most of the cases the assessed knowledge and skills are isolated and highly technical by their nature. Accordingly, an assessment primarily reflects the low cognitive level of students' ability at the level of memorization and procedures without connections. The revised approach calls for the assessment that involves the development of interdisciplinary knowledge and skills at a higher level of cognitive demand (e.g., understanding and reasoning).

From focusing on a single measure to a more diverse assessment. The traditional assessment is usually limited to a single test. Moreover, most of the tests used in the traditional assessment are multiple choice. As a rule, traditional tests measure single type of intelligence: for example, mathematics tests mostly measure logical-mathematical intelligence, language arts tests measure linguistic intelligence, etc. Moreover, the traditional assessment is mainly individual and does not include group assessment. The revised approach encourages diversity in assessment: variability of instruments, diversity of assessment methods, measurement of multiple intelligences, inclusion of individual and group assessment, etc.

From primarily quantitative to qualitative and integrated assessment. One of the key approaches is transition from primarily quantitative assessment to the assessment of multi-dimensional quantitative and qualitative characteristics of student learning. The traditional quantitative assessment does not always reflect real level of students' learning. Moreover, in some cases, it only provides a distorted estimate of where a student is in his learning curve. The quantitative assessment often overlooks such important characteristics of student learning as communication skills, ability to work in a team, level of student effort, individual style of thinking, etc. In order to evaluate these characteristics, assessment needs to include qualitative information through observations, interviews, analysis of student work, to name a few. The qualitative component of the assessment will greatly complement and enrich its quantitative component to provide a more comprehensive assessment of student's learning. Integration of quantitative and qualitative methods will help to make a shift from a "short-term" assessment of student's knowledge and skills to a "long-term" evaluation of student's intellectual potential as a learner.

From fixed and prescribed to more flexible assessment. The traditional system of assessment is strictly determined by policy regulations (standards, format, time factor, etc.). Of course, the traditional assessment policies have some advantages: they help to standardize assessment procedures and make it more objective. However, in traditional assessment procedures play a role of a "catcher": a student is punished for what s/he did not learn or did not know. At the same time, it shapes the type of mentality "what is assessed is what is valued". The revised projection of assessment acknowledges and evaluates everything that a student knows and can do in the subject domain even beyond the established programs and standards.

From standardized to open assessment. The standardized assessment is stressful. Studies have shown that under stress students cannot demonstrate even those knowledge and skills that they actually possess. A true assessment should be open and natural that relieves students' stress and tension. Moreover, open assessment should be connected to activities and projects that have personal, cultural and social relevance to students. Surveys, interviews, self-reflections, peer-assessment are some of the methods that could add certain level of openness and authenticity to traditional assessment.

From assessment to self-and-peer-assessment. In the traditional assessment, the answer to the question "who is the judge?" is easy; of course, it is the teacher who controls the assessment. The teacher evaluates student's achievements and reveals gaps in student's knowledge. The revised projection approach of the assessment recognizes student's self-and-peer-assessment (Sadler and Good, 2006). Moreover, the teacher's role as a judge is transformed to the role of a consultant, facilitator and advocate. Interaction between teacher and student is a natural extension of collaboration in achieving desired learning outcomes.

The revised dimensions emphasize characteristics of authentic assessment (e.g., continuity, diversity, integration, flexibility, openness, self-and-peer-assessment) that could facilitate effective learning environments in both face-to-face and on-line education.

One of the forms of such innovative authentic assessment is a learning portfolio (Johnson and Rose, 1997, Tierney, Carter and Desai, 1991) or e-portfolio in case of online learning (Challis, 2005). In general sense, a learning portfolio is a form of organization and a process of collecting, reporting, and analyzing the products of student learning. An e-portfolio (or e-folio) is defined as "a digitized collection of artifacts including demonstrations, resources, and accomplishments that represent an individual, group, or institution" (Lorenzo & Ittelson, 2005). Some scholars characterize a learning portfolio as a collection/exhibition of student work that comprehensively demonstrates not only academic achievements, but also the effort put forward by the student as well as the demonstrated progress in achieving learning outcomes (Arter, 1990; De Fina,1992). The portfolio could also be considered as a focused, systematic and continuous evaluation and student self-assessment of learning outcomes (Tierney et al., 1991). The main purpose of the learning portfolio is to showcase the student learning.

What is included in the portfolio? First, there is no clear list of items that should be included in a portfolio. This entirely depends on the requirements of a particular course/teacher. Second, the range of materials to be included in the portfolio also depends on learning objectives and expected outcomes of the course. Third, composition of the portfolio could also be restricted by certain items for inclusion. In general, a learning portfolio could consist of student work and evidence collected by a student and the documents submitted by others including teachers, peers, parents, etc. (Arter, 1990; De Fina, 1992; Johnson and Rose, 1997). The scope of items of a portfolio can be broad: samples of projects; student's independent work; homework; group work; presentations; essays; critiques; annotated bibliographies; literature review; student autobiography; journal log; collection of media resources (e.g., photographs, videos, sites, online encyclopedias) related to the subject domain; graphic work; spreadsheets; laboratory work; student resume; extracurricular activities; awards; etc.

Along with the student work, the learning portfolio could include evidence from teachers, classmates, parents, etc. This list could be also as long as defined by course requirements. Samples might include: teacher observations; evidence of teacher-student communication (e.g., emails, interviews, conversations); attendance checklist; copies of teacher's notes to parents, the list of assessments and teacher's comments on student work; the letters of support/reference from classmates, parents, community organizations; etc.

Aside from the student work and evidence from others, it is highly desirable that the portfolio includes the following essential elements: a title page; a cover letter describing the purpose and the brief composition of the portfolio; the content of the portfolio with the list of its main elements; reflection statement. This provides potential readers of the portfolio with a structure and makes it customer-friendly.

Some scholars advise to use two types of learning portfolios: a working portfolio and a final portfolio. The working portfolio could be considered as a depository of all student work that s/he produces during a term (a quarter or a semester). Further, a student could select from the working portfolio those items that were either required by a teacher or considered by the student as best evidence of her/his effort and progress — to be included in the final portfolio. In case of items selected by the students for the inclusion in the final portfolio, s/he can make sticky notes (physical or digital) on the margins of the best samples indicating — "my best work", "my favorite project", "the best thought-provoking article", etc. The teacher could also select from the working portfolio additional items that s/he considers original, interesting and deserving merit.

Technological advances allow composing and presenting a portfolio in a digital format through open sources specially designed for e-portfolios such as, for example, mahara.org (Figure 15).

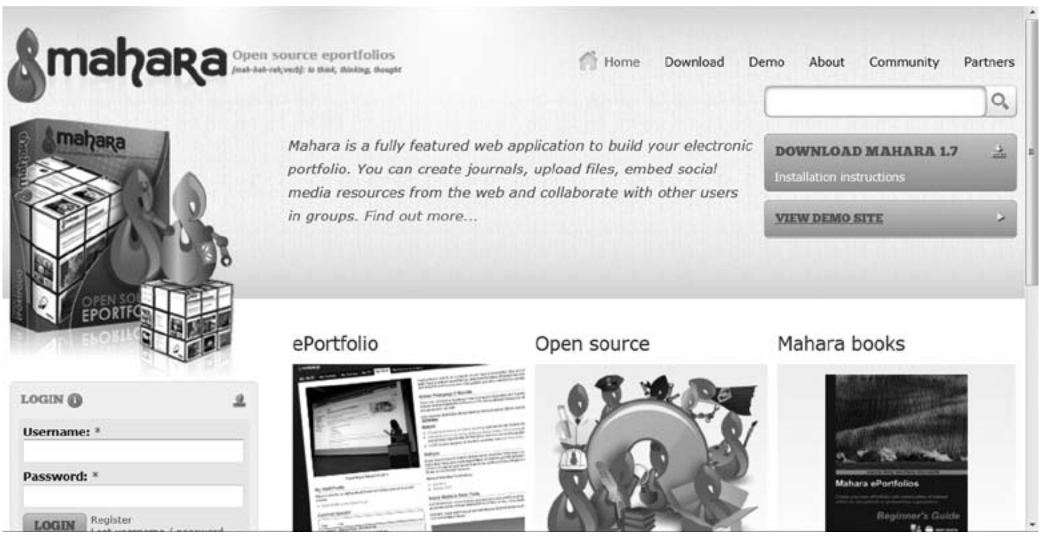

Fig. 15. Open source website (mahara.org) for the e-portfolio development

In some U.S. teacher training colleges during the final year of study, future teachers can use the e-portfolio developed as a requirement of their coursework for the employment purposes.

How learning portfolios can be assessed? The issue of portfolio assessment is rather complicated. First, the mandatory minimum and the optional maximum of items included in the portfolio should be clearly defined. Second, the rubrics, as well as the distribution of "weight" between the elements of the portfolio, should also be explicitly formulated. Third, summative score on the portfolio elements should be eventually converted to the grading system used in the institution.

Challis (2005) presents "a checklist" for a high quality e-portfolio that includes the following criteria: selection of material; level of students' reflection; content of the e-portfolio; use of multimedia; design of the e-portfolio; and customer-friendly navigation. Each criterion is described further through the list of specific descriptors. For instance, the criterion of selection of materials for e-portfolio requires it to be relevant (e.g., everything should be related to the purpose and audience) and thoughtfully structured (e.g., each point/example/illustration makes a useful contribution). The criterion of reflection should reveal the depth of students' understanding, self-awareness, growth, and responsiveness. The criterion of content should reflect the depth and breadth of thinking, as well as contextualization and personalization. It also reflects how accurately the content of the portfolio is and succinctly written and polished. The criterion of multimedia use addresses how the selected digital resources enhance the content and engage a reader, how appropriate and purposeful is the selection, and how well it is integrated into the e-portfolio. The criterion of design evaluates whether the portfolio is uncluttered and elegant, well organized and coherent. The criterion of navigation requires that the portfolio is fully hyperlinked, clear, intuitive, and allows users to select their own pathways (Challis, 2005). An example of a grading rubric for portfolio assessment is presented below.

An excellent portfolio is characterized by comprehensiveness; it meets the main criteria on material selection, level of reflection, use of multimedia, and navigation. The content of the portfolio shows student's consistent effort and significant progress toward achieving desirable learning outcomes. The content and design of the portfolio demonstrate quality, creativity, originality, and ingenuity. The navigation is clear and intuitive.

A good portfolio shows student's significant effort and progress toward major learning objectives. However, some of the criteria are not fully met. The content and design of the portfolio demonstrate the certain level of quality without distinct creativity and originality.

An average portfolio contains evidence of student satisfactory progress through the course. The content and design of the portfolio are limited to the required elements without distinct quality, creativity, and originality. Most of the criteria are not fully met.

A poor portfolio does not contain enough information to judge on student's satisfactory progress through the course. The content and design of the portfolio show limited or no effort to demonstrate its quality and originality. The navigation is poor.

Let us summarize the main benefits of using learning portfolios in education. In contrast to the traditional approach, which usually separates teaching, learning, and assessment, a portfolio attempts to integrate these major components. A portfolio allows for using both quantitative and qualitative methods of assessment through evaluation and analysis of student products and demonstrated effort. Portfolio approach encourages using multiple ways of assessing student performance, including teacher, peer and self-assessment. Portfolio promotes teacher-student collaboration in assessment. A portfolio could be used as a form of continuous assessment of learning progress that goes beyond a particular course. Finally, a portfolio can be easily integrated into the professional portfolio for employment and job seeking purposes.

Chapter 4

Engineering of Content

This chapter deals with the following main issues:

- modular design and content development;
- content interactivity and content communication;
- engineering of online learning.

4.1. Modular Design and Content Development

Engineering of content includes three major components:

1) content development;
2) content interactivity, and
3) content communication.

The content development is a the key element in the engineering of learning and it involves the process of planning and designing a variety of learning materials such as modules, activities, readings, discussions, assignments, assessments, and other items to support learning objectives of the course. Modular design is one of the most effective approaches used in the content development, particularly, in the design of online learning content.

Modular design. More broadly, modular design or modularity is defined as "an approach that subdivides a system into smaller parts (modules) that can be independently created and then used in different systems to drive multiple functionalities" (Clark and Baldwin, 2000). A modular design can be characterized by the following features: partitioning into reusable units consisting of isolated, functional, and self-contained elements; well-defined entry and exit characteristics of a module; flexibility in revision, change, and replacement of a module as needed. Modularity stands as one of the basic principles of the general systems theory. The principle of modularity determines the dynamics and the mobility of the system. Moreover, the system itself can be represented as a set of modules or treated as a separate module in the structure of a larger system.

The idea of using modular design in education was initiated in the 1970-ies (Goldschmidt and Goldschmidt 1974; Russell, 1974; and others) to provide self-paced individualized instruction. Scholars have outlined distinctive features of the modular design: breakdown of the learning content into 'manageable' parts (modules); screening of the content in order to eliminate 'extras'; maximizing self-paced individualized learning. A learning module is defined as "a self-contained independent unit of a planned series of learning activities designed to help students accomplish certain well-defined objectives" (Goldsmith & Goldsmith, 1973, p: 16) and it usually consists of learning objectives, content and activities, skill practice, and assessment. Sometimes module is also called a unit. A module may take several class periods, lessons, or, in some cases, several weeks to complete.

The traditional modular structure is limited and may include the following elements in a module: list of learning objectives, text (usually, in PDF format), and a test. In other cases, a module may include presentation (usually, in PowerPoint format), screencast, or video clip. The composition of elements in the module may also vary. The key question is whether and how module's structure supports student learning. This was a critical question in developing module's structure in the author's previous work (Tchoshanov 1996, 2011). We believe that module's structure and its elements should reflect the guiding principles of learning: building on students' prior knowledge, developing students' procedural fluency in a conceptual framework, and engaging students in metacognitive and self-reflective activities (Donovan & Bransford, 2005). Therefore, there are three main elements in a module: introductory elements, core elements, and applied elements as presented in Figure 16 (Tchoshanov, 1996). Pre-assessment and introductory elements address students' prior knowledge. Module's core and some applied elements aim at developing students' mathematical proficiency and procedural fluency within a conceptual framework. Applied elements such as generalization and debugging also focus on fostering metacognitive and self-monitoring skills.

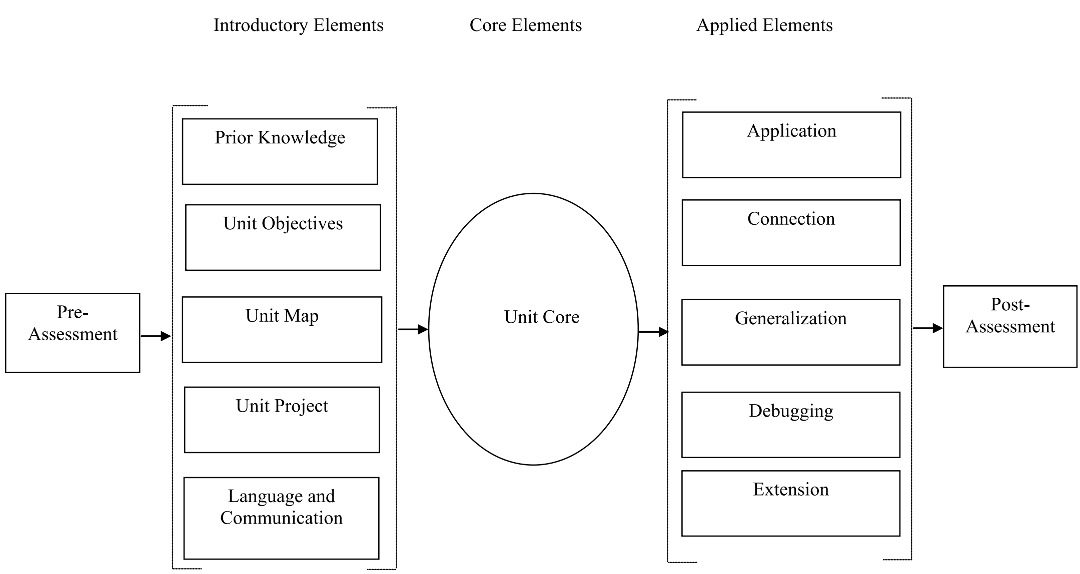

Fig. 16. The structure of a module/unit

Let us briefly describe each element of a module/unit. The pre-assessment element is designed to evaluate and determine student's readiness to learn the content of the unit. The introductory elements prepare students to learn the unit core. The unit objectives identify what students should know, understand and be able to do upon completion of the unit. The prior knowledge element consists of review of students' common preconceptions, skills, and experiences that may impact the learning of the unit core. The unit map provides a holistic picture of the content to be learned in the unit and represents the learning pathway through the unit. The unit project addresses the evolution and origins of fundamental concepts and ideas in the unit; it may include inquiry-type, problem-based explorations. The language and communication element identifies and clarifies main mathematical notions and concepts, and supports vocabulary development of English Language Learners (ELL). The unit core addresses the key concepts of the unit. A critical part of planning the unit core is sequencing major concepts to construct student's learning pathway. Applied elements aim at developing students' procedural and conceptual fluency of the content learned in the unit core. Application and connection elements are designed to enable students to practice skills in solving routine and non-routine problems and to deepen their understanding of the concepts. Generalization element provides a summative representation of the key concepts of the unit. Debugging element addresses common student's misconceptions, as well as ways to fix the misconceptions. Extension provides enrichment activities to further deepen student's knowledge. Post-assessment element is designed to evaluate overall retention of the module content.

Below we describe the unit planning and development activity at the University of Texas at El Paso (USA) using as an example one of the fundamental topics in in-service training of middle school mathematics teachers — Proportionality. The unit planning started with composition of the *didactical engineering team,* which consisted of five members with different expertise, including two college mathematics professors, one mathematics educator, and two mathematics coaches, whose primary responsibility was to work with teachers from local school districts. The team carefully conducted main steps in the didactical engineering process including analysis, design, and construction of the unit.

Analysis. *The m*ost important step was to conduct a comprehensive analysis of different resources, including but not limited to the literature review on issues related to learning sciences, modular design, proportional reasoning; analysis of state teacher standards and competences, as well as standards for middle school mathematics; analysis of assessment data and student work, etc. As a result of the analysis, the team understood that proportional reasoning is one of the challenging topics in the middle school curriculum. Moreover, most of the middle school teachers were not sufficiently prepared to teach it.

An analysis of student work is a critical step in the engineering of learning. This analysis allowed the team to use the idea of model tracing discussed in Chapter 3.2. (Koedinger and Corbett, 2006) to identify common student misconceptions and trace different possible actions students may take in solving proportionality problems. One of the most common student misconceptions is applying additive reasoning in proportional situations. For example, in solving the following 'mixture' problem: "Would three cups of iced tea and two packets of sugar be sweeter or the same as four cups of iced tea and three packets of sugar?", most of the middle school students (grades 7 and 8) would use the idea of constant difference to justify their answer (Figure 17). Student's response "It's the same thing because both ice tea cups would have one cup without sugar" clearly indicates the 'additive reasoning' misconception.

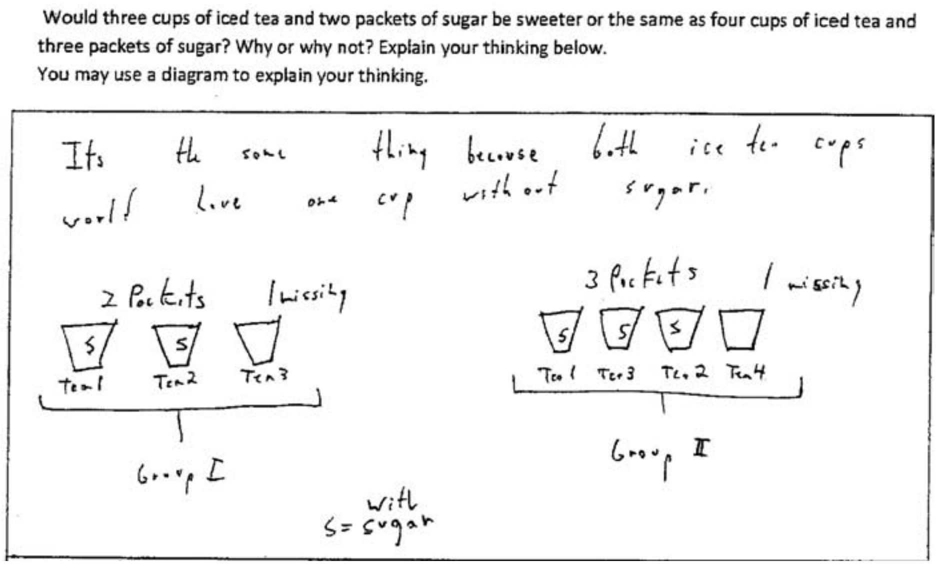

Fig. 17. An example of misconception in applying additive reasoning in proportional situation

The same misconception is observed in another proportional situation with a slightly different context. When students were asked to solve the following 'lever' problem: "The lever below holds a 3-unit weight at a distance of 3 units from the fulcrum (center) — on left side, and a 2-unit weight at a distance of 4 units from the fulcrum (center) — on the right. Is the lever balanced?" A sample of student work with the same misconception of applying additive reasoning ("it even[s] out if you *add* 4 and 2 also 3 & 3") in a proportional situation is presented in Figure 18 below.

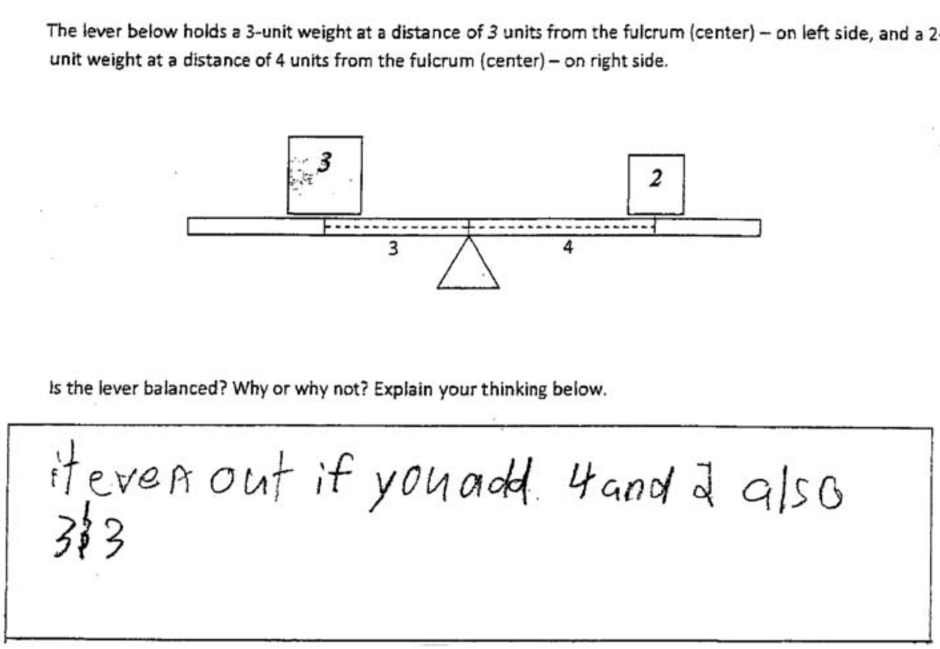

Fig. 18. "Additive reasoning" misconception in the lever context

Two samples of student work presented above illustrate how strong the additive reasoning misconception could be. These types of misconceptions should be carefully considered in designing the learning pathway and constructing the unit and its elements.

During the analysis stage, the team also studied advances in learning sciences and modular design. The team decided to use the modular design in connection with guiding principles of learning to construct teaching products. The team used a set of key questions that needed to be addressed in the process of planning and developing a unit (modified from Brahier, 2005):

- What students are expected to know and be able to do by the end of the unit? (Unit objectives)
- What type of experiences have students already had with this topic? (Prior knowledge)
- As students explore the unit, what are the key concepts and skills they will encounter and need to understand? (Unit map and core)
- In what order should the key concepts be learned? (Unit core — sequencing)
- How many lessons will it take to accomplish the learning objectives for the unit? (Unit core — timing)
- What kind of experiences should students have to help them learn these concepts? (Unit core and applied elements)
- What materials and tools will support learning this unit? (Resources)
- How do you know if students are ready for the unit? At the end of the unit, how do you know whether students really understand the most important concepts in the unit? (Pre-and post-assessment)
- Once students complete the unit, what is the next logical step in the students' learning sequence? (Inter-unit planning)

Design. The most critical question for the team at this stage was — what was the learning pathway to proportional reasoning? In order to conceptualize the learning pathway, the team spent several working sessions trying to find answers to the key questions: what is a starting point in the development of students' proportional reasoning, what constitutes an end point, and how to get from one to another? Analysis of resources helped the team to realize the importance of student transition through the following conceptual corridor: "rational numbers — proportional relationship — direct variation model — linear function", where the concept of "relationship" plays a central role. Moreover, the team discussed potential learning outcomes — what are key characteristics of a good proportional thinker? The team came up with the list of characteristics, which later were converted into learning objectives, activities, and learning outcomes:

- To know and understand the idea of relationship;
- To recognize multiplicative situations and distinguish them from additive situations;

- To recognize and explain the difference between proportional ($y = mx$) and non-proportional ($y = mx + b$) situations; it was important to note that in the latter situation, y was not proportional to x, but rather, Δy was proportional to Δx;
- To recognize and distinguish between different types of proportionality: direct and inverse;
- To use proportionality as a mathematical model in real-world contexts;
- To know and use the language of proportionality;
- To understand the concept of function to express the co-variation;
- To recognize that the graph of a direct proportional situation $y=kx$ was a straight line that passed through the origin;
- To recognize that the graph of a non-proportional situation was a straight line intersecting the y axis b units away from the origin;
- To know that the graph of an inversely proportional situation was a hyperbola;
- To understand that k was the constant *ratio* in *direct* proportional situations;
- To understand that k was the constant *product* in *inverse* proportional situations.

Based on the list of characteristics, the team designed the following learning pathway to the proportional reasoning presented in Figure 19.

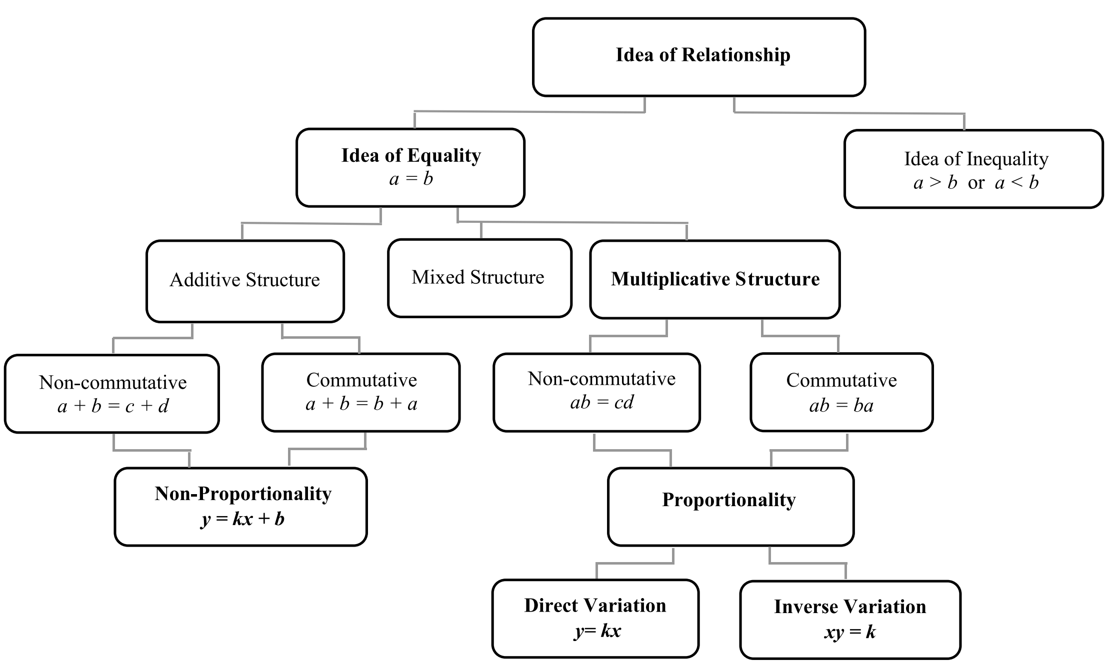

Fig. 19. Learning pathway to proportional reasoning as a unit map

Along with the learning pathway, the team discussed another key design question — what constituted an effective learning? Effective and ineffective learning may start with either no-conception or pre-conception on student side. If the learning is ineffective, the student preconception is converted to misconception and then, if no debugging takes place, it leads to a mistake and eventually to misunderstanding. Effective learning is engineered by building on student preconception and further strengthening it through the development of student's conception, concept, and understanding. Ineffective learning could become effective, if the debugging is built into the learning process through recognizing student misconception, addressing it and continuously supporting student's self-monitoring (Figure 20).

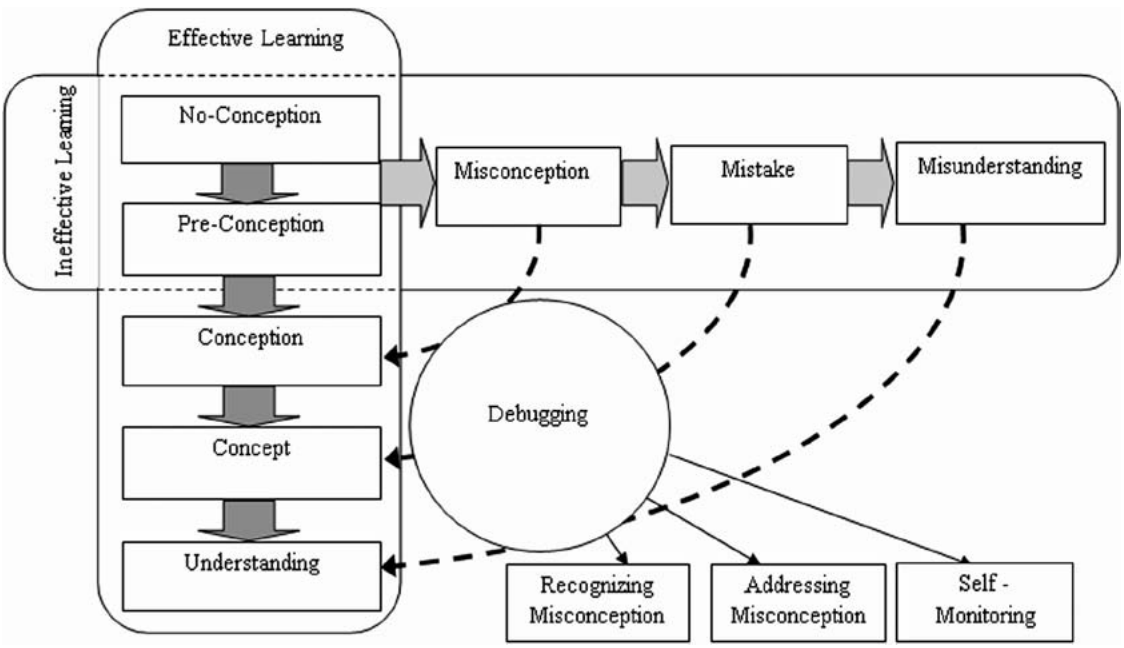

Fig. 20. Effective vs. ineffective learning design

Construction. Having analyzed and designed the learning pathway, the team modified the conceptual corridor: "relationship — multiplicative structure — proportionality — direct variation — linear function" to start constructing teaching products — the unit and its elements. The team selected the "Mission Atlantis" as the unit theme. At the construction stage, the team also considered appropriate ICT resources to support student learning. Blackboard LMS (Learning Management System) (see Figure 21) and the open source software Geogebra were chosen as primary digital technologies for the implementation of the unit.

Fig. 21. Screenshot of the course "Learning Pathway to Proportional Reasoning"

Based on the list of characteristics of a proportional thinker, the team performed a backward design and conducted an inventory of prior knowledge and prerequisite skills that students would need in order to work on the introductory elements of the unit. Then the inventory was used to design *pre-assessment* to evaluate and determine students' readiness to learn the content of the unit.

The list of characteristics of a proportional thinker also helped the team to select and develop the *unit objectives* to identify what students should know, understand and be able to do upon studying the unit. After the successful completion of this unit, the student was expected to:

- apply mathematical process standards to use in proportional and non-proportional relationships to develop foundational concepts of functions;
- represent linear proportional situations with tables, graphs, and equations in the form of $y = kx$;
- represent linear non-proportional situations with tables, graphs, and equations in the form of $y = mx + b$, where $b \neq 0$;
- contrast bivariate sets of data that suggested the linear relationship with bivariate sets of data that had not suggested a linear relationship from a graphical representation;
- use a trend line that approximated the linear relationship between bivariate sets of data to make predictions;
- solve problems involving direct variation;
- distinguish between proportional and non-proportional situations using tables, graphs, and equations in the form $y = kx$ or $y = mx + b$, where $b \neq 0$.

In order to construct the *prior knowledge* element, the team analyzed students' common preconceptions that may impact the learning of the unit core. One of the common preconceptions is a lack of understanding of a difference between a notion of conversion and a concept of relationship. The team developed a case study to address this preconception.

Making Sense of Dollars and Pesos[4]*.* This summer, Andy saved money to buy a guitar. He waited for the right time to ask his mom to go across the U. S. border with him to the Juarez city market. He took his summer savings, $132 dollars, and they headed to Mexico. When they arrived at the market, Andy learned that the cost of Guitaras Valencianas was 15,000 pesos. He did not have enough money and had to borrow money from his mom to buy the guitar. How much money, in U.S. dollars, had to borrow Andy? Try the following:

1. express "1 U. S. dollar equal to 10.77 Mexican pesos" in ratio form
2. use your ratio to set up a proportion showing this relationship, use D for dollars and P for pesos
3. determine the correct number for the following: 1 Mexican peso =___U. S. dollars
4. use the statement in the previous question to set up a proportion showing this relationship, use D for dollars and P for pesos.

Next, answer the questions:

1. What difficulties, if any, did you have in working on the problems a) — d) above?
2. What did you have to remember to work on the problems b) and d)?

Ms. Benning was going over a new unit for her eighth grade mathematics class. She wanted to assess what her students had understood and remembered about ratio and proportions from the seventh grade. Also she wanted to have an activity that would engage and interest her students. In her eighth grade mathematics class, Ms. Benning's plan was to have students start with proportions and extend the concept to direct variation, a concept in mathematics that modeled various everyday phenomena. She usually taught the eighth graders taking Algebra, but because of her students' success with the performance on last year's state tests, she was asked to teach a class of 8^{th} graders who had not scored very well as 7^{th} graders. About one third of the students were English language learners, about half of the class had scored right at or a few points below the passing score in the 7^{th} grade assessment, and only five students were progressing well enough to be ready to take algebra as ninth graders successfully next year.

Ms. Benning knew she was being challenged with this very diverse group of students. Also, from the past experience she knew some students never learned how to work proficiently with rational numbers, which was essential in understanding proportionality. She wanted to prepare these students not just for the state exams but also for their 9^{th} grade Algebra course, which was essential in understanding Mathematics and a major challenge for most ninth graders.

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[4] This case study was written by L. Michal and edited by M. Tchoshanov.

At the start of class, Ms. Benning was careful to engage every student. She first asked, "Who has recently used the exchange rate from dollars to pesos to buy something at across the U. S. border at Mercado Juarez?" Many of her students raised their hands and wanted to share some of their experiences. Because she only had 46 minutes for class, she was careful to select a few students but really wanted to hear from everyone. Carla was an English language learner and rarely shared any comments with the class, so Ms. Benning was pleased to see Carla raise her hand. She asked Carla to share her experience. Carla shared that they spent most of the day at the Centro Artesanal, the city's new Art center, with her family from Phoenix. Charles, on the other hand, said his family had taken his Uncle Bill to the market and had gone to the bank the day before to change dollars to pesos. Ms. Benning asked Charles, if he knew the current exchange rate. Charles said "I think one dollar is equal to about eleven pesos." The two, Carla and Charles, were completely different students. Ms. Benning saw them both on the same plane mathematically but not on the same plane in class. She wished there would be a way to get Carla to share more with peers, because even though her language was limited, her knowledge of Mathematics surpassed that of all the other students in the class.

Ms. Benning asked the students to form groups of 2 to work together in pairs. She proceeded with the idea of having students see how one set of numbers generated another set of numbers. To see how much her students had remembered from the seventh grade, she proceeded with her plan to have students work in pairs to get them to use the exchange rate to uncover their understanding of the topics and concepts which had been covered in the seventh grade mathematics. As she walked around the room, she noticed Nolan, the school football star, hiding the morning newspaper under his notebook. "Nolan, do you have the morning newspaper with you?" Nolan was caught completely off guard and said, "Uh, yes, Ms. Benning, but I…""Good," said Ms. Benning, "will you turn to the front page of the business section and write the exchange rate between U. S. dollars and Mexican pesos on the blackboard?" Nolan checked the rate and wrote 1 U. S. dollar = 10.77 Mexican pesos. "Thanks, Nolan. Alright, for the sake of simplicity let us round off the pesos to the nearest peso." Ms. Benning knew this was something most students should know by now but wanted to see how many students remembered this. Nolan felt he had to contribute an answer and asked, "Is it ten point seven eight?" To confront Nolan's response, Ms. Benning asked Adriana to share her answer and explain how she got it. Adriana said, "We got 11 pesos. First we saw the number after the decimal point, saw that it was seven and so changed it to 11." "Okay, good" said Ms. Benning, "what did you change to eleven and also why did you change it? You are right; I just want to make sure others can see what your thoughts were?" Adriana said, "Um… we saw seven after the decimal point, and since seven is more than 5, we changed the 10 to 11."

Ms. Benning restated what she thought Adriana had wanted to say, "So you changed the 0 to 1 which made the 10 and 11, right?" Adriana nodded, yes. "Okay, does everybody remember how to round off? If not, please see me for some problems to work before tomorrow." Ms. Benning reminded them to work in pairs and went back to the equation Nolan had written on the board "1 U. S. dollar = 11 Mexican pesos" and said, "If we write this equation in a ratio form, what would this look like?" She selected David to come up to the board. "Okay, David, can you write the ratio on the board?" David wrote "1:11" on the board. "Okay, good. Carolina, I noticed you had another way to write the ratio, will you write it on the board for us?"

Carolina wrote, "$\frac{1}{11}$" on the board. "Great, so from your work in Mathematics class you remember that a ratio can be written in two ways. We will use the second way to write the ratio and build on that form. Are there any questions before we go on?" Rodrigo raised his hand and asked, "Could we also write eleven over one?"

"Yes, that is another way to write the ratio. We just have to remember to associate 11 with pesos and 1 with dollars," said Ms. Benning.

Ms. Benning was getting ready to go on to proportions, when someone said, "could we also write 11 pesos over 1 dollar?" Ms. Benning said, "Yes, however, when we write the number with the unit of measurement, the ratio becomes a rate. So, when writing a ratio with denominate numerals, numbers with units of measurements, you are writing rates. I sense that you have remembered proportions and we are going to do next."

Before doing proportions, Ms. Benning asked the students to write an equation in numerical symbols. Jerry came up to the board and wrote, "D = 11P." Ms. Benning asked the class to use what Jerry had written to change 5 dollars into pesos. Jerry was the first to ask, "Do we use the 5 where the D is?" After this question Ms. Benning suggested that students set up a proportion with using the ratio and the variables D for dollars and P for pesos. Several students offered answers to this and she finally wrote on the board, "$\frac{D}{P} = \frac{1}{11}$". "Okay, let's use this proportion to see how many pesos we have when you have dollars. Let's use a table to organize our work and then plot these pairs of numbers on a rectangular coordinate system, remember to use D for dollars and P for pesos. Please note the table has a process column in the middle for you to record what you are doing each time to determine the amount of pesos." Ms. Benning continued to walk around the class to see the work of students in pairs. The students had an activity sheet with the table to organize their work and a rectangular coordinate system to plot their ordered pairs. She saw the students filling in the table without filling out the process column. She added again "When you offer a value for P, be prepared to tell us how you had found the value, so use the process column to write down what you are doing to get P."

After giving the students a few minutes to fill in the table of values, she asked for volunteers to give their values for P and also how they had determined those values. Carla was ready to give the first answer. "Ms. Benning, for 5 dollars, there are 55 pesos. I multiplied 5 by 11, but if I had 5 pesos, would I divide by 11?" It took Ms. Benning by surprise, so she did not have an answer ready. Instead of answering, she asked the class to answer. She said, "A very good question. Do we divide by 11, class?" This created a lot of noise in the classroom with students talking about it, so Ms. Benning was pleased that she had deferred the question to the class. As she was getting ready to answer the question, the bell for the next period rang and she ran out of time to summarize what they had done. She felt very good about the "noise" at the end of class time, and would be prepared to start the next day with the answer for Carla.

Questions for the case study analysis and discussion:

1. What was Ms. Benning's main purpose in her lesson on exchange rates?
2. Did the class work let her get at what students understood and remembered about proportions?
3. Were the needs of the English language learners addressed in class?
4. Did she deal with the case of Nolan and the newspaper appropriately?
5. How would you have answered the last question that was asked in class?
6. What did she learn about student learning?

Later, the case study was converted into the pre-unit activity that teachers could use in their own teaching (the *pre-unit activity* is presented in the condensed version of the unit in the Appendix).

Studies show that more than 80% of college students have similar misconception (Lochhead and Mestre, 1988). Indeed, in situations such as the conversion at the center of this activity (11 pesos makes 1 dollar), a very common student misconception, called the reversal error, is to translate the conversion "11 pesos = = 1 dollar" to 11P = 1D; but if we interpret that equation so that P stands for the number of pesos, and D stands for the number of dollars, then the correct equation is in fact 11D = 1P. Research (PCK Tools: Students' Understanding of Symbolic Representation, Consortium for Policy Research in Education (CPRE), 2003) also shows that this is a tough misconception to fix. One reason is that 11P = 1D is what students get when they "translate" the words directly to symbols. The word by word translation of a problem is usually effective in conversion statements, but does not work here because it uses P and D as the units of quantities, instead of the quantities themselves. Put another way, this misconception is interpreting 11P as "11 pesos" instead of "11 times the number of pesos". In order to fix this misconception we ask students to express the currency exchange physically (e.g., "place one dollar on the table . . . ") and verbally (e.g., "describe how you would tell someone to exchange pesos to dollars"). After in-depth discussion on the case study, the team decided to further address the common misconception and constructed an algorithm for translating words into Algebra to help teachers to distinguish between conversion statements and algebraic relationships, between labels and variables.

Step 1. Identify the quantity you are dealing with. For instance, consider the problem "One green square tile consists of four red square tiles. Let G be an area of the green square tile and R be an area of the red square tile. Write down a symbolic representation of the relationship between G and R". In this problem, the quantity we are dealing with is area. Correspondingly the problem addresses the *conversion* between G and R, which represent *labels* for different areas.

In contrast, consider the problem "One green square tile consists of four red square tiles. Let G be the number of the green square tiles and R be the number of the red square tiles. Write down a symbolic representation of the relationship between G and R". In this problem, the quantity we are dealing with is a number. And, correspondingly the problem addresses the *relationship* between G and R, which represent *variables*.

Step 2. Probe your guess/solution for reasonableness. Ask yourself questions: which tile has a greater area? Are there more red tiles than green tiles in a given shape?

Step 3. Write the symbolic representations: for the conversion statement (*G=4R)* and for the algebraic relationship (*R=4G)*.

Step 4. Check your symbolic representation by plugging in several numerical values. Does the representation make sense?

In the first problem, the conversion statement considering the area of the red tile as 1 unit square, it makes sense that the area of one green tile would be equal to the area of four red tiles. In the second problem, the algebraic relationship: if we plug in 2 instead of number of the green rods (G), we get R=8, which makes sense (there are 8 red tiles in 2 green tiles).

Let us illustrate the above algorithm with another example: One gallon is equal to four quarts. Let us denote gallons by G and quarts by Q. Write down: (1) a conversion statement between G and Q; (2) an algebraic formula for the relationship between G and Q.

The difference between different types of symbolic representation in case of conversion statement and algebraic relationship is depicted in Table 7.

Table 7. Comparison between conversion statement and algebraic representation

<table>
<tr><th rowspan="2">Symbolization</th><th colspan="2">Type of Symbolic Representation</th></tr>
<tr><th>Conversion Statement</th><th>Algebraic Relationship</th></tr>
<tr><td>Role of Symbols</td><td>Labels</td><td>Variables</td></tr>
<tr><td>Quantity</td><td>Capacity</td><td>Number of Units</td></tr>
<tr><td>Process of Symbolic Representation</td><td>Direct Translation of the Statement to Symbols

1 Gallon = 4 Quarts</td><td>Proportion/ Table $\frac{G}{Q} = \frac{1}{4}$
<table><tr><th>Gallons (G)</th><th>Quarts (Q)</th></tr><tr><td>1</td><td>4</td></tr><tr><td>2</td><td>8</td></tr><tr><td>3</td><td>12</td></tr><tr><td>...</td><td>...</td></tr><tr><td>G</td><td>4G</td></tr></table>Q = 4G</td></tr>
<tr><td>Symbolic Representation</td><td>*1G = 4Q*</td><td>*Q = 4G*</td></tr>
<tr><td>Written Representation</td><td>"One gallon is equal to four quarts"</td><td>"For every gallon there are four quarts"</td></tr>
</table>

The importance of the *unit map* (Figure 19) in representing students' learning pathway to proportional reasoning was discussed earlier in the design section.

As mentioned above, the *unit project* addresses the evolution and origins of the fundamental idea of the unit — multiplicative relationship. The team used a project based on one of the famous historical discovery: Archimedes, a Greek mathematician, was the first to explain the principle of lever. Although he did not prove this principle, he was the first to state, "weights of equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance". When weights are equal, distances of the weights from the fulcrum must be adjusted to have a balanced state of equilibrium. Therefore, Archimedes lever principle tells us, if $W_1D_1 = W_2D_2$, then the above is in static equilibrium, with all torques balanced. The distance from the point where you place the weight W_1 to the fulcrum is the lever arm distance D_1 and the distance from the point where you place weight W_2 to the fulcrum is the lever arm distance D_2. Distances are measured from the fulcrum to weights. Archimedes is said to have stated, "give me a lever and a place to stand and I can move the earth". Imagine you have a huge lever on one side of which you have the Earth and on the other side a place to stand. How far would you have to stand to move the earth!? The unit project is included in the Appendix.

The language and communication element identifies and clarifies mathematical vocabulary as well as describes how a student will acquire the content knowledge using different modalities (listening, speaking, reading, and writing). This element may include some of the following language objectives: explicitly teach the vocabulary required to master the content objective; include a description of the interaction, in which students will participate, such as discussions or paired and/or cooperative learning activities; give students the opportunity to use functional language—reading, speaking, listening, and writing—in the content area; encourage reflection at the end of lesson to assess whether or not language objectives were met; and encourage teacher to model the behavior that supports the student expectation (adapted from Texas Education Agency, 2006). For the unit on proportionality, the team selected the following language objectives: use mathematical vocabulary to explain orally or in writing the main properties of proportionality; construct the Venn diagram to contrast and compare proportional and non-proportional situations; create a list of attributes of proportional and non-proportional situations; explain how to solve a proportion to a partner; describe the relationship between diameter and circumference of a circle; write, in your own words, an explanation of the proportionality concept; connect an informal language to the formal mathematical language in a graphic organizer; write out the formulas that are related to proportionality; engage in a "Walk About" or "Gallery Walk" activity ("What have you learnt today?"); and construct a flip book of vocabulary words on the topic of proportionality with representative diagrams.

The u*nit core* addresses the key concepts of proportionality and it is a critical element in constructing a logical sequence of activities to support student's learning pathway. The team included the following activities united by the theme "Mission Atlantis" in the core (missions are presented in the Appendix):

- Mission One: Representing proportional relationships;
- Mission Two: Proportional relationships;
- Mission Three: Non-proportional relationships;
- Mission Four: Proportional and non-proportional relationships.

Let us briefly consider the content of the first mission. It consists of the following activity — "Besides astronauts, the International Space Station, ISS, has also hosted tourists from planet Earth. Between 2001 and 2007, five tourists have traveled to ISS at an average cost of $25 million per person. Currently, 200 seats for tourists have been presold" — and a set of guiding questions listed below:

1. Complete the table to show the cost of different numbers of tourists if the cost remains the same. Let t represent the number of tourists and c represent the cost of tours in millions.
2. Write how you would find the cost of 11 tours.
3. Explain the rule you would use to find the cost of any number of tours (t).
4. Graph the relationship between the number of tourists and the cost of tours. You may use Geogebra (Figure 22) to construct your graph.

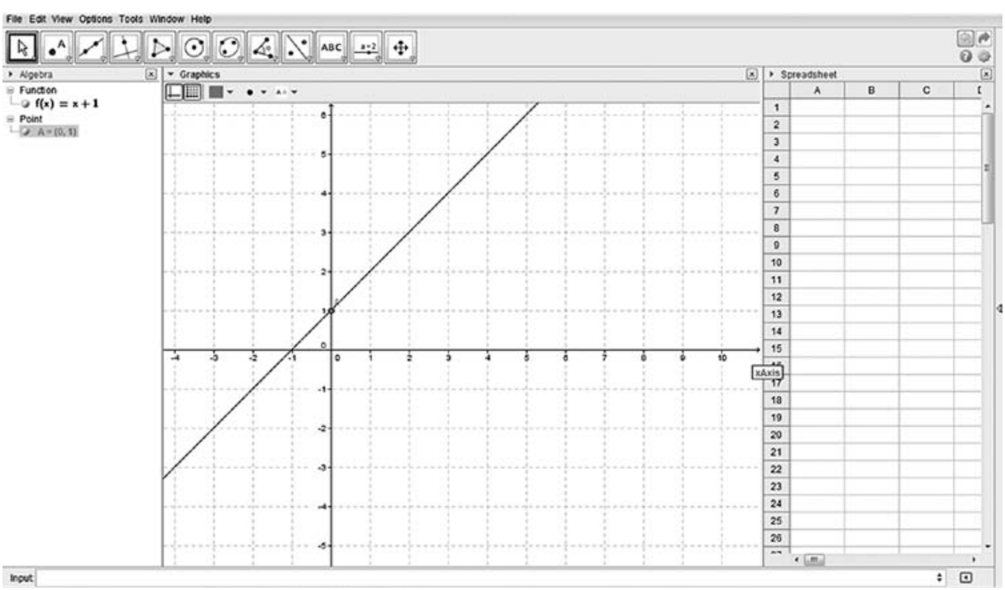

Fig. 22. Screenshot of open source mathematical software Geogebra

5. Use your rule to write an equation that describes the relationship between the number of tourists, t, and the cost of tours, c.
6. What is the rate of tourists to cost? What is the rate of cost to tourists? What is the unit rate of cost to tours?

7. What will the cost be for 200 tourists? Use a proportion and the equation to verify this cost.
8. How many tourists would be able to travel for $425 million? Show your work.
9. Identify where the unit rate appears in the table, the graph, the equation, and the proportion.

Each mission is supplemented with instructional notes and mathematical discussions[5]. The instructional notes for the "Mission One", for instance, describe the planning parameters for teaching the activity and include:

- *Time schedule*: 1-2 days
- *Vocabulary*: Relationship, Rate, Unit rate
- *Materials and ICT resources*: Geogebra
- *Teaching method*: Primarily individual and small groups, with a class discussion at the end, when discussing where the rates appear in the equation, graph, and proportion. There could also be a brief discussion at Question 3 ("Explain the rule you would use to find the cost of any number of tours.")

The mathematical discussion addresses the main goals of the activity: compare the table, graph, and equation for the simple proportional relationship. In particular, students should see the rate, or constant of proportionality, in each of these representations of the relationship. The tasks for this activity will be repeated, with constants of proportionality that are increasingly harder to describe, in subsequent activities.

The mathematical discussion also describes important terms that deserve a closer consideration. For instance, what is the difference between ratio and rate? Lamon (2007) states that "Early definitions of ratio and rate were linked to comparisons within and between measurement spaces. A ratio was considered a comparison between like quantities (e.g., pounds : pounds) and a rate — a comparison of unlike quantities (e.g., distance : time), although, as noted by Lesh, Post, and Behr (1987), there was "a disagreement about the essential characteristics" that distinguished the two." Below is the list of the definitions of key terms the team used throughout the unit:

- Relationship is an equality, inequality, or any property for two objects in a specified order, for example $a=b$, $a<b$, $ab=ba$, etc.
- Ratio is a multiplicative comparison between two quantities.
- Rate is "reflectively abstracted constant ratio" (Thompson, 1994). *Unit rate* is a rate with the second term equal to one. *Composite unit* is a constant ratio expressed in the lowest terms.
- Slope is the constant rate of change for the related linear function.
- A relationship between two varying quantities is proportional when the ratio of corresponding terms is constant.

5 The mathematical discussion section was written by A. Duval and edited by M. Tchoshanov.

The mathematical discussion also includes "tricky points" that students may have difficulty with, for instance, the concept and representation of the quantity *millions*. There are two points of view about the use of "millions" in the cost. The computationally simplest way is to interpret "million" as the part of the unit (along with dollars), and treat the "numerical value" of the cost per tourist as 25. The alternative is to treat the "numerical value" as 25,000,000, but all extra zeros can get awkward. There is nothing wrong with the point of view, as long as we are consistent and careful to not lose the units since the units are very important to all rate problems.

Another tricky issue is the point (0, 0). As with the balance activity, the point (0; 0) is included in our relationship, since the relationship is proportional. Here, though, it may be easier to understand; if no tourists, it does not cost anything. It's still a good idea to point out how (0, 0) shows up in the table and the graph, and satisfies the formula. In the proportion, it is less clear how (0, 0) works. For instance, we do not want to plug in 0 and 0 into the proportion, because we would have 0/0, which is indeterminate. Not undefined, as some students and teachers might think. The further discussion on the difference between undefined and indeterminate could be worthwhile. Let $x \neq 0$, then $x/0$ is *undefined,* because if you try to define it as $x/0=k$, you end up with a contradiction: $x=k\times 0=0$. However, let $x=0$, then $0/0$ is *indeterminate,* because if you try to determine it as $0/0=k$, you have a statement $0=k\times 0$, which is true for any k.

Last but not the least, the mathematical discussion includes an overview and clarification of some activity questions as listed below.

Question 1: "Complete the table…" For this question and the next one, students might use the formula ("$25 million per person"), or eventually notice that each entry in the table is $25 million more than the previous entry. These are both important points of view.

Question 4: "Graph the relationship…" Once again, students should notice that the data forms a straight line that passes through the origin.

Question 6: The unit rate question. Describe the unit rate; it should be easy in this activity where the proportional relationship was initially defined in terms of a unit rate.

Question 7: "Use a proportion and the equation to verify this cost." In the "Using a Proportion" column, students should verify the solution they had got (probably, with the equation 200 tourists x $25 million= cost). Alternatively, some students may set up a proportion to solve the problem, but then they should verify their answer with an equation: 200 tourists x $25 million/tourist= $5000 million. A minor teaching opportunity: either way, the equation is a good time to highlight how units and "dimensional analysis" work.

Question 9: "Identify where the rates appear in the equation, the graph, the table, and the proportion." This question captures the main goal of the Mission One activity. The Mission One, as well as other three missions, is included in the unit (see Appendix).

Applied elements are constructed to assist students in developing their procedural fluency and conceptual understanding of proportional reasoning. *Application and connection* elements are designed to provide students with practice skills in solving routine and non-routine problems and to deepen students' understanding of the proportionality concept. An example of the application element is enclosed in the Appendix.

In order to assist teachers with assessing students' proportional reasoning, the following levels of proportional thinking were used by the team (adapted from Langrall and Swafford, 2000).

Level 1 — Non-proportional reasoning: a student guesses and/or uses visual clues; is heavily dependent on additive structures and unable to recognize multiplicative relationships; randomly uses numbers, operations, or procedural strategies; is not capable to link two measures and establish relationship between them.

Level 2 — Pre-proportional reasoning: a student uses pictures, models, or manipulatives to make sense of situations; makes qualitative comparisons; uses repeated addition to solve proportional situations.

Level 3 — Quantitative proportional reasoning: a student unitizes or uses composite units; finds and uses unit rate; identifies or uses the scale factor or a table; uses equivalent fractions; builds up both measures.

Level 4 — Symbolic proportional reasoning: a student sets up a proportion using variables, understands the meaning of the proportion and solves it using a cross-product rule or equivalent fractions; fully understands that the ratio between two values stays constant even though the values themselves may change.

The *Debugging element* deals with common student's misconceptions as well as ways to address misconceptions. Debugging is represented in the table with the following columns: type of misconception, example of misconception, cause of misconception, and way to fix the misconception. In constructing this element, we identify three main groups of errors/mistakes: epistemological, methodological, and common student errors. *Epistemological mistakes* are made by scholars in the process of scientific evolution and historical development. They are caused by the relativity of human knowledge, its incompleteness and limitations. The use of epistemological errors in teaching and learning fosters students to think critically and substantially changes students' attitude: students no longer perceive science as a set of ready-handed knowledge, but rather as a historical drama of ideas, as an intellectual struggle of different schools of thought. Of a particular interest among the epistemological errors are the so-called "great" errors. Louis de Broglie, a prominent physicist, considered it very useful to reflect on the mistakes made by great minds, as they often had a good reason to make them. *Methodological errors* are errors of teaching: they occur when the teacher lacks the knowledge of content, didactics (including learning sciences) or both. *Common student errors* are typical mistakes that occur in learning. We group errors in debugging tables and use them as a means of improving student learning. In other words, if the traditional approach is limited to transitioning from student's preconception to new knowledge, in modular design we expand student's zone of proximal development (ZPD) to the zone of advanced/perspective development (ZAD) (Vygotsky, 1978) where students may 'self-fix' their errors. An example of the debugging element is enclosed in the Appendix.

The generalization element is included in the unit (see Appendix) to represent a synthesis of the key concepts of the unit and connections between the concepts. *Extension element* provides enrichment activities to further deepen students' knowledge and understanding of proportionality including both direct and inverse variations. In both cases of direct and inverse proportionality k is called *a constant of proportionality.* The key difference in the role of k between direct proportional and inverse proportional relationship is as follows: in the inverse proportional relationship $xy=k$, k represent *a coefficient;* whereas in the direct proportional relationship $y=kx$, k represents a coefficient with a special role — *a constant rate of change.* Extension element is enclosed in the unit (Appendix). *Post-assessment* element is designed to evaluate overall student learning and understanding of the proportional reasoning. Item analysis of student work on the post-assessment helps a teacher to identify the node in the student learning pathway and corresponding activity which needs to be repeated in order to successfully complete the unit.

Overall, the modular design has a number of advantages, in particular, its content and instructional flexibility (Tchoshanov, 2011). For instance, as modules are developed, they could be used in multiple courses and professional development workshops. Throughout the coursework an individual module could be maintained and updated separately without impacting other modules if needed. Moreover, modular design helps an instructor to select and appropriately sequence modules to meet the main course requirements. On a student part, it reduces a course to a set of key topics, simplifies the design of the course, and makes it easier to focus on major learning objectives. The modular approach could also optimize the course development through team-teaching and development when different modules are designed and taught by different instructors; at the same time, every instructor in the team could teach the entire course.

Among the weaknesses of the modular design the opponents (FESC, 1986; Russell, 1974) emphasize fragmentation of learning with the allocation of a large proportion of coursework to independent study mode; opponents see it as a lack of proper guidance; ignorance of the integrity and the logic of the subject; reduction of the course to a series of discrete and disconnected units; and challenges in designing modules.

As modular design is progressively utilized in the online content development, many of these shortcomings are gradually smoothed out. This particularly applies to preserve the integrity and the logic of the subject as well as addressing defragmentation of student learning through carefully designed conceptual corridors and learning pathways (Tchoshanov, 2011).

4.2. Content Interactivity and Content Communication

Along with content development, content interactivity and content communication play important role in the engineering of learning. In this section, we will consider some of the approaches that will help to enhance content interactivity, such as cognitive visualization and other emerging techniques, for example, video streaming, screencasting, and gamification. We will also discuss different formats of content communication in this section.

Visualization is one of the few areas of research in education, whose relevance is continuously increasing over time in different subject domains including Mathematics. It was relevant in 1957, when P. Van Hiele first presented the model of teaching Geometry with a support for the development of student visual thinking (Van Hiele, 1986). The relevance of this problem sustained in the 1970-ies, when R. Skemp proposed the theory of conceptual scheme (Skemp, 1987). The significance of the visualization problem was emphasized in the 1990-ies by the publication "Visualization in teaching mathematics" (Zimmerman and Cummingham, 1990). The level of relevance of this issue is still dominating nowadays with its critical role in designing content interactivity for online learning (Sigmar-Olaf and Keller, 2005; Konate, 2008).

The direct application of the science of learning' findings in visualization such as "People learn better from words and pictures than from words alone" (Mayer, 2011: 70) to the practice of learning through recommendation "Add relevant graphics to text lesson" (Ibid.: 70) sounds invigoratingly simplistic. The meaning of visualization in learning is much broader yet complex than just 'adding graphics to the text'. Moreover, visualization plays a significant role in the engineering of learning via linking advances in the science of learning and the practice of using visualization in the classroom as shown in Figure 23.

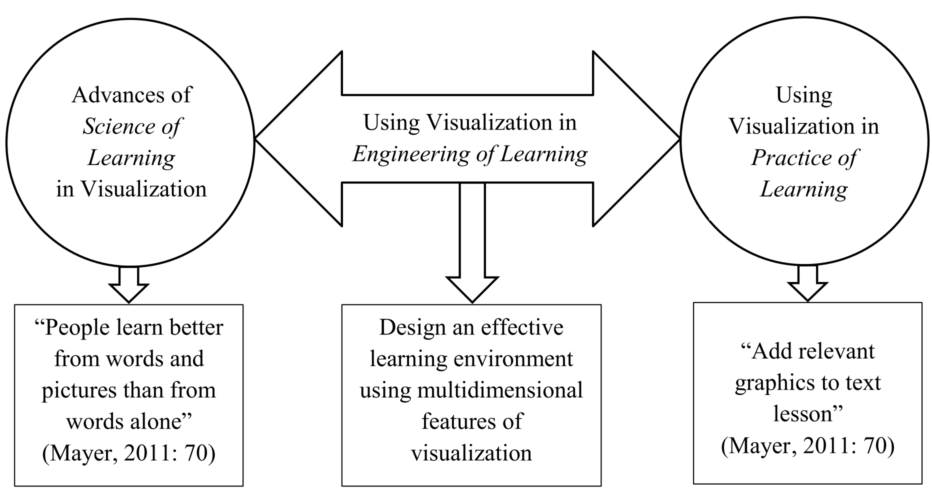

Fig. 23. Engineering of learning as a link between the science of learning and the practice of learning in using visualization

Visualization is a multidimensional construct that has several important characteristics. We will consider the following dimensions:

- illustrative and cognitive visualization
- static and dynamic visualization
- passive and interactive visualization
- isolated and connected visualization
- visualization and multiple representations
- academic and scientific visualization.

Visualization could be illustrative and cognitive. Illustrative visualization usually represents an answer to a low cognitive demand question such as: what is it? For instance, if one asks “what is an isosceles triangle?”, a visual illustration of a triangle with two congruent legs would be a sufficient answer. Cognitive visualization goes beyond just illustration: it unpacks the meaning of the concept. For example, cognitive visualization is used to develop students’ understanding of problem solving and proof in Mathematics. Let say, we would like to visually represent the proof of the following theorem “Sum of interior angles of a triangle is equal to a straight angle”. The proof of this basic theorem requires multiple steps, which are depicted in the cognitive visual representation (Figure 24).

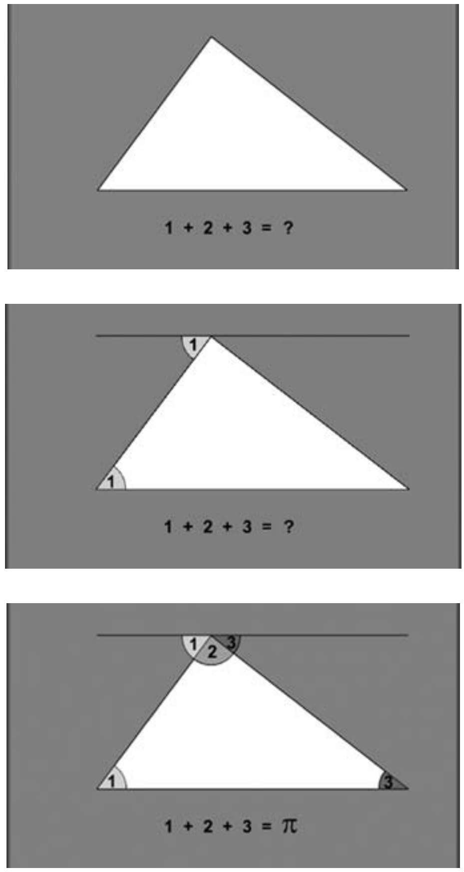

Fig. 24. Cognitive visualization of the theorem for sum of interior angles of a triangle

Visualization could be static and dynamic. Using the above example (Figure 24), we could represent the final step as a static visual image of the proof, or we could show the same proof in dynamics as a series of steps. Most of the visual proofs presented in a fascinating series "Proof without words: Exercises in visual thinking" (Nelsen 1993, 2000; Nelsen & Alsina 2006) are primarily static. Author's open access website on Visual Mathematics (http://mourat.utep.edu/vis_math/) consists of examples of cognitive dynamic visualization on various topics of Mathematics (Figure 25).

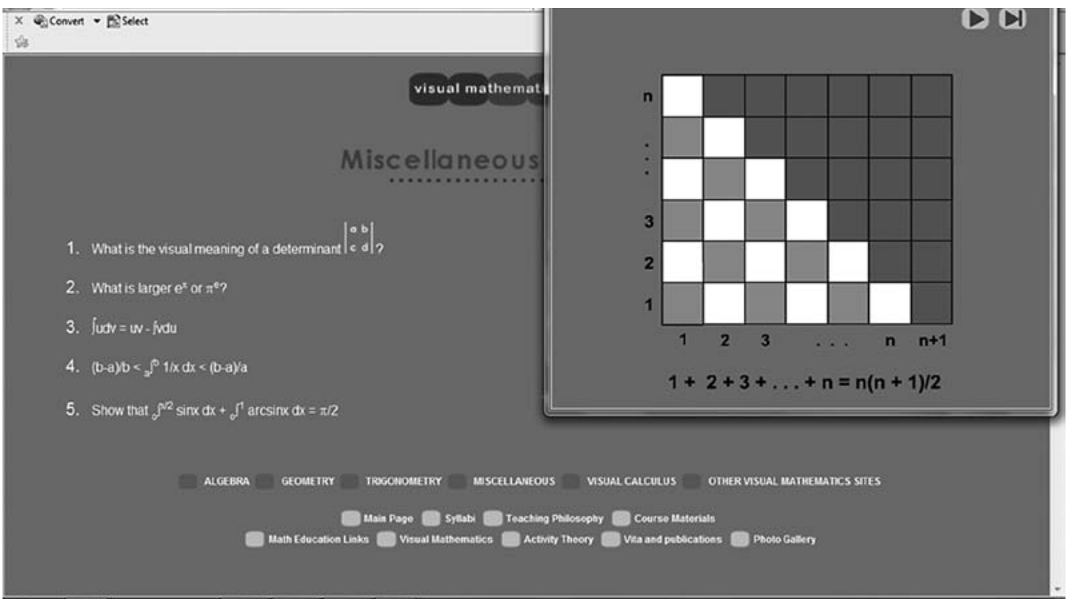

Fig. 25. Screenshot of the Visual Mathematics website

A dynamic visualization feature helps learners to develop their conceptual understanding and is intensively used in a variety of software packages such as Geogebra, Geometer's Sketchpad, Cabri, Mathematica, to name a few.

Visualization could be passive and interactive. Passive visualization requires little or no student involvement in the visualization process whereas interactive visualization allows students to manipulate certain parameters of the demonstration to better understand the concept. The open source Wolfram Demonstrations Project (Figure 26) presents interactive visual solutions using computer animations and applets to various mathematics and science problems where students can 'play' with the demonstration changing its parameters. For example, interactive visual solution to the problem of an area under cycloid presented in the Figure 26 has multiple benefits compared to an analytic solution: students can visually follow the trace of the cycloid, they can understand how the curve is produced, students can visualize the concept of the area under the cycloid, and finally, they can build conceptual understanding of why the area under the cycloid produced by a circle with a radius R is equal to $A = 3\pi R^2$.

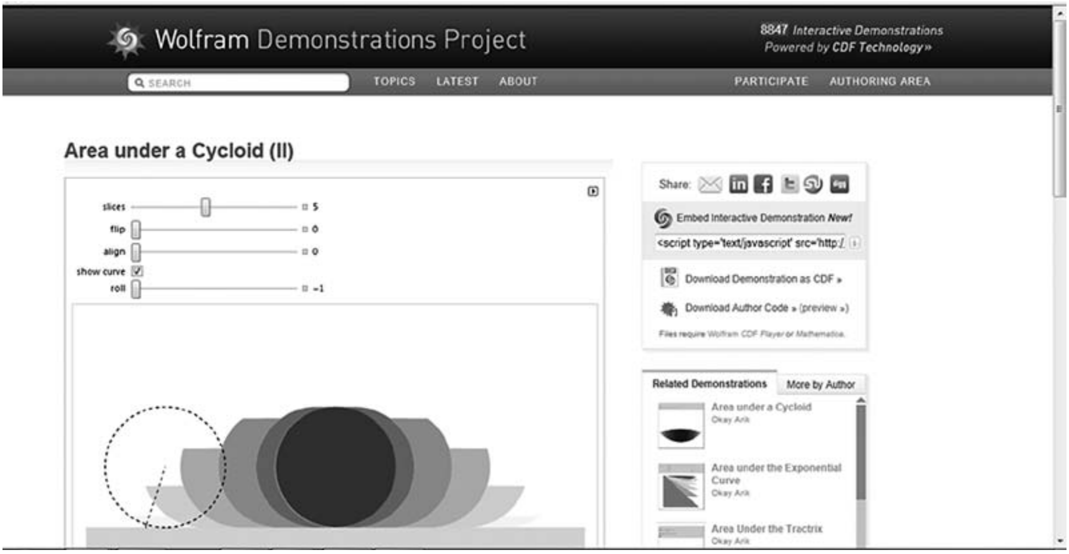

Fig. 26. Screenshot of the Wolfram Demonstrations Project

Visualization could be isolated and connected. Let us consider the following problem "The cookie monster sneaks into the kitchen and eats half of the cookie; on the second day he comes in and eats half of what remains of the cookie from the first day; on the third day he comes in and eats half of what remains from the second day. If the cookie monster continues this process for four days, how much of the cookies has he eaten? How much is left? If the process continues forever, will he ever eat all cookie?" The author used this problem in one of his graduate classes with in-service teachers while discussing possibilities of early introduction of the infinity concept at the middle school level. In order to look for the solution, teachers usually start with making a table with the values given in the problem. Very few of them use visualization as a problem solving tool. After the class discussion on different methods of solving the "Cookie Monster" problem, they admit that the visual solution is the best one in developing students' understanding of the concept. One of the possible visual solutions is shown in Figure 27.

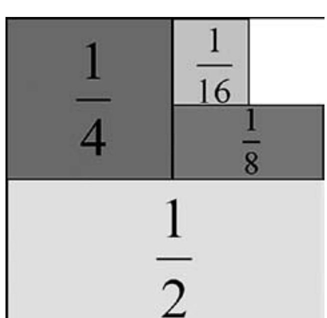

Fig. 27. Visual solution to the "Cookie Monster" problem

The discussion is further extended to other visual representations of the problem: teachers get engaged in considering the number line (using a bread stick instead of a square-shaped cookie), a pie model (using circle-shape crackers), or even cubic (using a 3D cubic-shape brownie) visual representation of solution. The teachers understand that within the same modality of visualization there could be multiple ways to represent the same concept. Most importantly, the teachers see the difference between an isolated visual image and multiple connected visual solutions for the same problem.

Visualization could be used as a singular mode and as one of the modalities in multiple representations. Using the same "Cookie Monster" problem, the teachers were able to synthesize multiple methods of solving the problem into the multiple representational diagram depicted in Figure 28. The visual solutions discussed above (e.g., number line, pie, square and cube models) are presented along with other multiple representational modalities (e.g., tables, graphs, equations, diagrams).

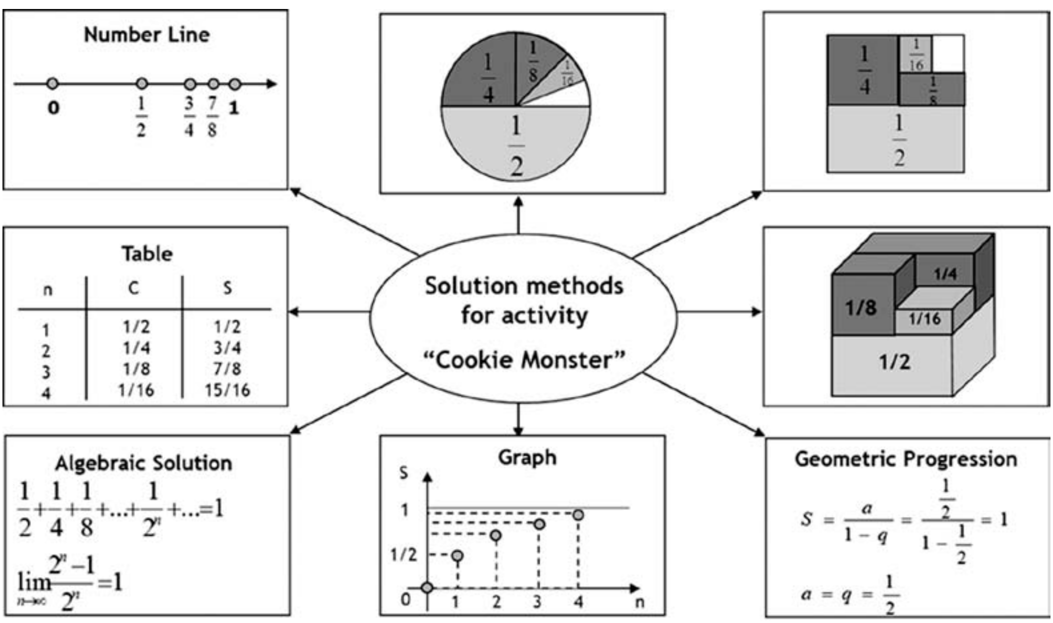

Fig. 28. Representational modalities for solutions to the "Cookie Monster" problem

Last but not least, visualization could be academic and scientific. The visualization examples presented above are all academic by nature because they are used to support student learning in a particular academic discipline. Scientific visualization is an interdisciplinary branch of science which is "recognized as important for understanding data, whether measured, sensed remotely or calculated" (Wright, 2007) and it is primarily concerned with visualization of three-dimensional phenomena in scientific research. Therefore, scientific visualization could be too advanced for students to grasp and understand. An important question here is how to get students motivated in searching for and appreciating the scientific visualization. For example, most of the high school and college students know what a 3-D cube looks like. However, many of them might be curious to know and surprised by what a 4-D cube looks like (see Figure 29: http://upload.wikimedia.org/wikipedia/commons/a/a2/Tesseract.ogv).

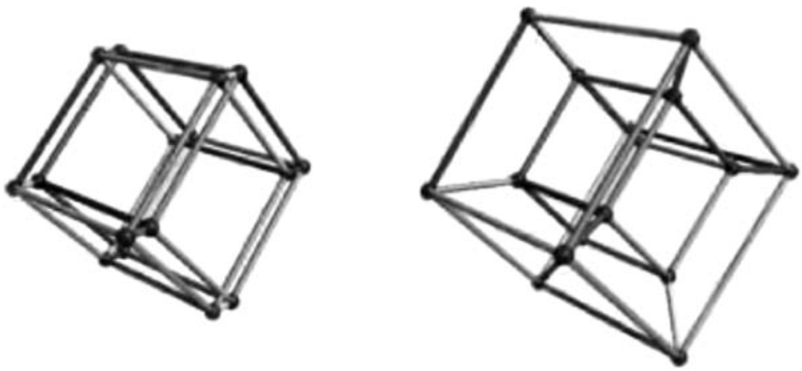

Fig. 29. Visualization of a 4-D cube: orthogonal (left) and perspective projection (right)

Addressing the visualization issue would be incomplete without considering the role of visual tools in the form of concept and/or mind maps to support student learning and understanding (Wycoff, 1991). The main purpose of a concept map is to engage students in making connections between concepts and procedures and expand students' understanding of a subject domain through a holistic perspective. An example of the concept map is presented in Figure 30 (http://www.svsu.edu/mathsci-center/uploads/math/gmconcept.htm).

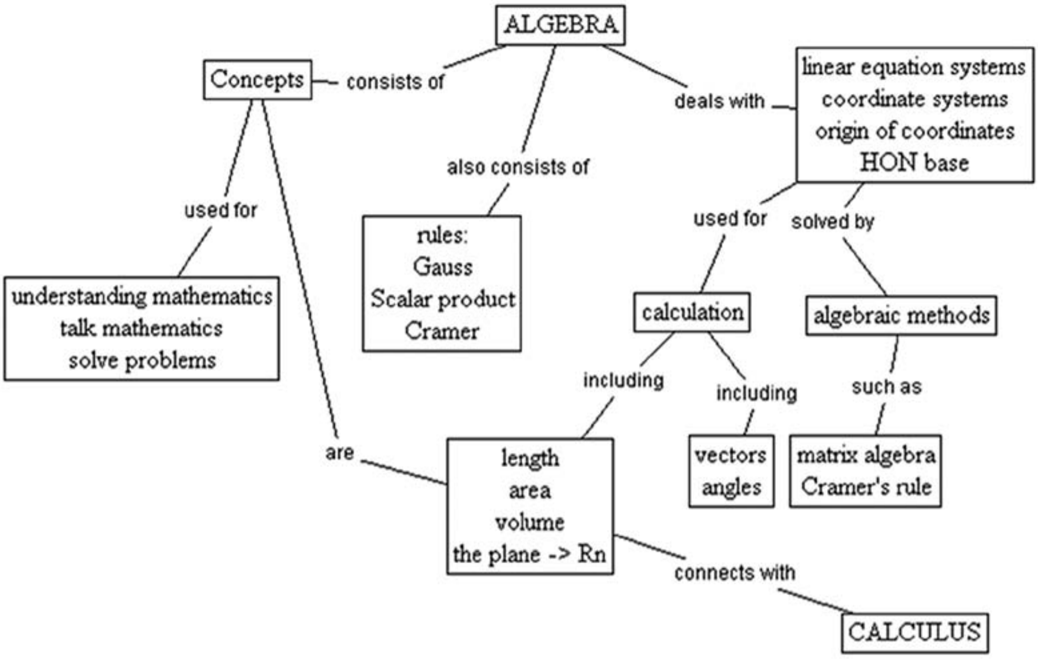

Fig. 30. Example of a concept map for Algebra

Video and/or media streaming is another widely used technique to enhance content interactivity. Video streaming helps learners to understand complex concepts that are not quite convincing to explain with plain text and graphics (Klass, 2003). Video streaming is particularly important for online learning due to its distinct interactivity component. Incorporation of multimedia including video streaming can improve the learning process as students see the concepts and ideas in action (Michelich, 2002). “In addition, a moving image can help students visualize a process or see how something works. Video can take tacit information or knowledge that may be too difficult to describe in text into an articulate, vivid description through the use of images” (Hartsell and Yuen, 2006: 32). Video streaming can evoke emotional reactions and increase student motivation. Furthermore, streamed videos can be accessed by students at any location that has an Internet access (such as library, home, café) and at any time. Another advantage is a student choice over priority and sequence of video materials to be observed on-demand. The true advantage of video streaming is an opportunity for self-pacing online learning: students are in charge of starting, pausing, skipping, and reviewing the media material. Among major limitations in implementation of video streaming in online learning could be resources, support structure and personnel training, since “it is difficult to sustain streaming video in academic institutions because of limited access to technology and knowledgeable experts who can assist maintaining and developing media streaming” (Shepard, 2004). There are ample opportunities for video and media streaming offered by variety of educational sources such as Discovery Education (http://streaming.discoveryeducation.com/), National Geographic (http://video.nationalgeographic.com/video/), NBC Learn (http://www.nbclearn.com/portal/site/learn/) and many other resources. An example of NBC Learn media streaming site on “Science of NHL Hockey” is presented in Figure 31.

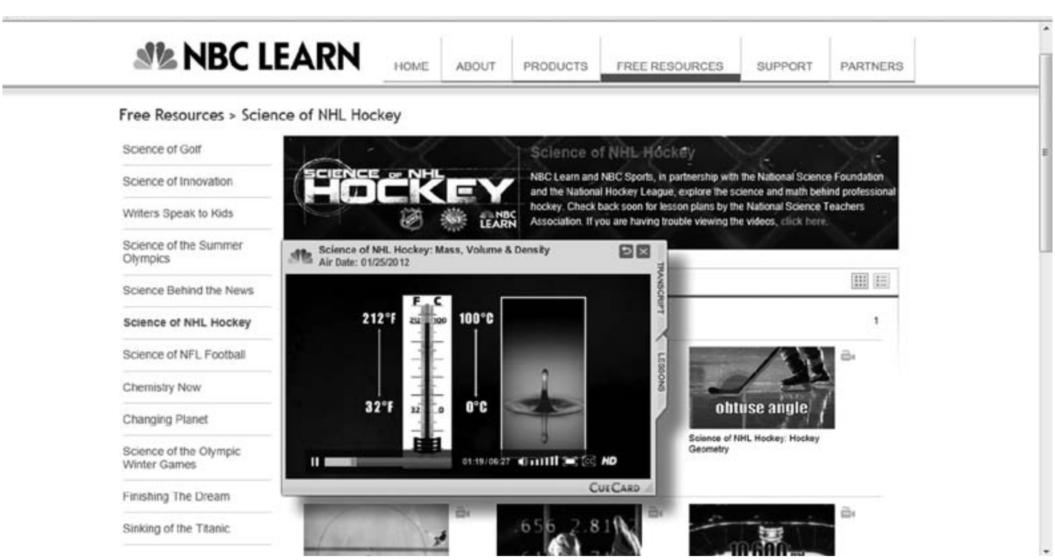

Fig. 31. Screenshot of the NBC Learn media streaming resource

Screencasting is a technique of creating dynamic and engaging content through digital video and audio recording of a computer screen while developing tutorials and demonstrations. Screencasting could also be used for digital storytelling and narrated presentations with a variety of media (e.g., video clips, pictures, graphs, and animations) imported into it. There are multiple advantages both for students and instructors in incorporating screencasting in learning. Screencasting is an effective tool that helps teachers to explain difficult concepts and allows students to learn a sequence of steps in performing a certain procedure, working on a task and solving a problem. Similarly, with video streaming, students can watch a screencast anywhere and anytime. Moreover, students can review any part of the screencast, pause, rewind, and repeat it as needed, which creates an effective learning environment for self-paced learning. Screencasting can be used to fulfill a variety of learning objectives, including but not limited to topic introduction, overview of the concept, discussion, and skill practice. Screencasting is widely used by open source repositories, such as Khan Academy (Figure 32), to provide opportunities for "*flipped classroom*" activities (Bergmann & Sams, 2012) when students watch teacher's screencast lecture as a homework and use class time for discussing difficult topics and challenging problems, working on projects, activities, etc. In order to produce a quality screencast, teachers need to have screencasting software (e.g., Webinaria, Jing, Screencast-o-Matic) and the screencasting tools such as microphone (for narration), webcam (for video), digital tablet or touch-screen with stylus (for drawing), etc. "The most obvious drawback of screencasting is that it is not interactive. Although some lessons lend themselves to fixed demonstration, others do not and should not be taught with screencasts… Simply recording the instructor's screen during a class session can be an inefficient way to transfer information" (ELI, 2006).

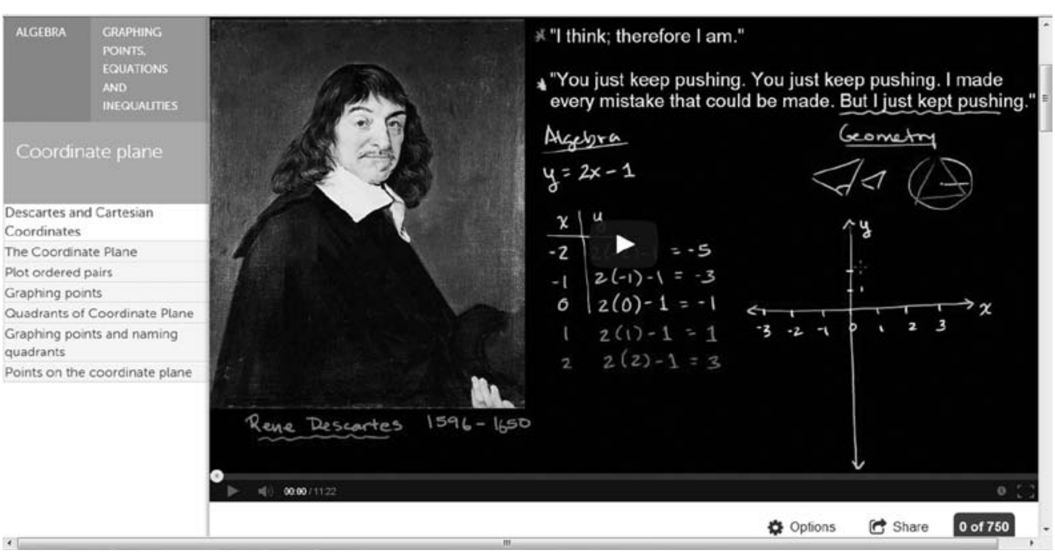

Fig. 32. Screenshot of the Khan Academy use of screencasting

Gamification, game-based learning, or game-informed learning are the names for the emerging phenomenon in education — "using game-based mechanics, aesthetics and game thinking to engage people, motivate action, promote learning, and solve problems" (Kapp, 2012: 10). As a pedagogical approach, gamification is constructive by nature and built on the elements of multiple intelligences' theories, situated learning, experiential learning and the activity theory. Gamification allows students to learn and experiment in a non-threatening environment, supports learning by doing through social interaction and collaboration. Gee (2007) emphasizes that "a good instructional game would pick its domain of authentic professionalism well, intelligently select the skills and knowledge to be distributed, build in a related value system as integral to gameplay, and clearly relate any explicit instructions to specific contexts and situations".

Well-designed gamification has multiple benefits including but not limited to providing authentic learning context and activities, multiple roles and perspectives in co-construction of knowledge as well as encouraging scaffolding and integrated assessment. An example of gamification is "Function game" where by inputs and outputs you have to identify a function (Figure 33).

Along with benefits there are some limitations to the gamification approach. The *content should be a major driving force for designing game-based learning.* Unfortunately, gamification based on the quiz-and-reward format only is not the most effective way to engineer learning and motivate students. Well-designed gamification supports high cognitive demand content and focuses on students' understanding and reasoning more than just memorizing facts and procedures. Another critical consideration in gamification has a natural and seamless connection between the game and the learning: the game improves the learning and the learning supports the game. A well-designed gamification also carefully balances content, learning, and assessment.

Fig. 33. Screenshot of the "Function game"

Content communication. Along with the content development and content interactivity, promoting and facilitating content-focused communication between the instructor and students is critically important to the success of the course whether it is face-to-face, hybrid, or online. With regard to distance learning, the content communication is an essential point of distinction between truly effective online course and poorly designed old-fashioned correspondence course. The content communication within an online course could be organized in individualized and/or group-based format. It also could be synchronous and/or asynchronous. Regardless of the format, the communication is a key to creating and sustaining an effective learning environment in the course.

In order to initiate and encourage communication between students, it is helpful to provide an opportunity for students to introduce each other at the beginning of the course. There are various tools available to support individualized communication such as texting, e-mailing, using Skype, FaceTime, Facebook, Twitter, etc. Instructor may choose to schedule phone or Skype conversations with individual students in an online course during virtual office hours which should be posted in the course syllabus. As an instructor of the course, you may also interact with individual students via text messaging and e-mailing. Another form of virtual communication with individual students is using Skype and/or FaceTime that enables face-to-face interaction by video as well as by voice. Instructor may also use social networking tools such as Facebook and/or Twitter to communicate with individual students as well as with the groups of students and the whole class through posting messages, blogs, and other ways of promoting communication.

Group communication and discussions are equally critical for the online course as individual communication. Various learning management systems offer multiple channels for group communication such as chat rooms, different modifications of discussion boards (e.g., Contribute, WebEx), collaborative document sharing and editing tools in real time (e.g., Google Docs, CampusPack). These virtual tools allow students and the instructor to engage in a text-based synchronous group conversation and discussion for various purposes including but not limited to the review sessions for major course assignments, to discuss group projects and presentations. Instructors have preferences in using particular tools for the group communication. Let us share an example of using the Blackboard discussion board to promote group communication in a content-specific topic. The graduate class of in-service middle school teachers was assigned to read the chapter on rational numbers and take a test. One of the questions in the test is below:

"Which of the statements below is true?

a) $2.4999\ldots < 2.5$
b) $2.4999\ldots = 2.5$
c) $2.4999\ldots > 2.5$
d) Cannot be determined given the above information.

Explain your answer."

The level of complexity of this item is determined by its connection to the fundamental idea of duality. Most of the class participants felt unfamiliar and challenged by the question posted in the assignment. Some of the students who selected the answer "a", e-mailed the instructor expressing the confusion. The most trivial solution to this situation is that the instructor could simply provide a correct answer to 'avoid' discussion on the challenging concept. However, this option would significantly limit student learning. The instructor (his signature in the Table 8 is represented as *mt*) decided to provoke the whole class discussion using the Blackboard. As depicted in the table, the discussion consists of four major stages:

1. Provoke: instructor selects a provoking question and invites participants to the discussion; the instructor monitors student responses and provides clarification.
2. Sustain: instructor capitalizes on students' reasoning to require further exploration.
3. Evaluate: instructor asks students to explain and evaluate the solution.
4. Synthesize: instructor brings a closure to the discussion.

The table also includes discussion actions and discussion context to illustrate the complexity and challenges of purposefully-orchestrated discussion in supporting student learning[6].

Table 8. The fragment of content communication via discussion board

Discussion stage	Discussion action	Discussion context
Provoke	Instructor selects a provoking question and invites participants to the discussion	*Dear All, one of the participants had difficulty understanding the problem 5 on Chapter Test #3. The student wrote: "I don't understand why my answer (letter A) was incorrect. 2.4999... has to be smaller than 2.5". Do we have people answering this problem differently? Share your responses, please. mt* *Dr. Tchoshanov, I agree with the student, due to the construct or the limited information though of the quesiton regarding the answer responses. I understand what the student is thinking. 2.4999 is smaller than 2.5, unless you estimate the value (though this was 'not' indicated as an approximation). They are "virtually" the same, but they are not, there is a difference which is miniscule. There is no way we could view the difference. For example, in measurement all measurements are approximations, a measurement of 2.5 and 2.4999... would be virtually the same, if you are in 'approximation.' Techinally, it is smaller value even if the value is a miniscule in difference. Brianna* *Brianna, I also agree with you. Mathematically, I think 2.4999.... is less than 2.5 because there is a very small difference in between these numbers. Also, we can say 2.4999... is approximately equal to 2.5. I do not think 2.4999.... is equal to 2.5. If we see this problem through student's point of vie, 2.4999... is equal to 2.5. Because, in a number line, 2.4999.... is very close to 2.5. We teach them to round to the nearest number in the number line. Pat* *When I answered this question I was picturing a number line which in that case the 2.49999 is smaller than 2.5, but then I second guessed myself thinking should I round up to the nearest tenth? If so, the two numbers would be equal. I guess as you say it all deals with the approximity of your numbers. Enrique*

[6] Students' grammar and style intentionally have been left unchanged.

Discussion stage	Discussion action	Discussion context
		I too think that if you look at it in a technical and mathematical way, 2.4999 is literally smaller than 2.5, but if it is being compared through the form of approximation then they are the same. Depends on how you look at it. Radhika *Radhika, I completely agree on your thoughts, it really depends how you are viewing the contexts of this problem. I do not believe there was sufficient amount to answer if greater than or equal. It does depend on how you see it, I do not think it incorrect. I put D. for the answer (I view things in a technical light) since all the above answers is plausible, if your counting the approximations or not. Good point. Brianna*
	Instructor monitors student responses and provides clarification	*However, the problem didn't ask for rounding or approximation. mt* *I think we can all make a strong point for every answer choice there was, but the question did not state if this was an approximation or not, so i read the question in its most literal definition and chose the answer the was most correct, I also chose A. Jaime* *I agree that it really depends on how you view it which is why I also chose D on this question. I can definitely see why A looks like a good answer because really it could be true but I too think it depended on how you viewed the problem which is why I ultimately chose D. Samantha* *When I answered this question, I chose to think of it in terms of fractions. For instance, 1/3 can be represented physically. But if you put it in decimal form, 1/3 is the same as 0.3333.... Then I thought to myself, is this number less than 0.34? Yes! I can represent both. So to me 2.4999.... is less than 2.5. I as well do not understand why a is wrong. I went through the reading as well as searched the web and looked in my old math texts. I did not find anything contradicting my idea. Ann*
Sustain	Instructor capitalizes on students' reasoning to require students exploring further	*… let me provide you with a counterexample to sustain the discussion. Ann uses a very convincing argument saying "1/3 is the same as 0.333..." If we accept Ann's argument, then let's do the following:* *a) lets multiply both sides of 1/3 = .333... by 3;* *b) (1/3)x3=(.333...)x3* *c) 1=.999...!* *Share your insights on 1=0.999..., please. mt* *Dr. Tchoshanov, lets consider the inequality that we use for domain and range of a function (introduction of function in Algebra 1) with a graph using closed and open circles. For example, the domain of a graph of a function with an open circle at x=1 extend to the negative infinity is -∞≤x<1. Even though the function is very close to x = 1, the domain is not -∞≤x≤1. Thank you. Rick* *Rick, very valid point. Thank you. The question is how do we connect the two ways of reasoning about the same concept? mt* *… I asked a middle school math teacher and she didn't know. Then I asked an engineer and he sent me this email:* *Debbie,* *2.49999... = 2.5. To prove this, assume: 10 * x — x = 9 * x, so: 24.9999... — 2.49999... = 9 * 2.4999...Considering that 0.0999... will cancel in the subtraction, then: 24.9 — 2.4 = 9 * 2.4999... Simplifying: 22.5 = 9 * 2.4999... Dividing by 9: 2.5 = 2.4999... QED* *It did make sense. We know that simply substituting numbers didn't necessarily make something true. Here is a case where you could try simple numbers like two or three and the final numbers would be the same, but if you substituted 2.4999..., it would come out as 2.5 on one side and 2.4999... on the other. However, the expression still holds even though there is a case where substituting doesn't work. This is a very interesting problem and I'm curious to see what others will say about it. Debra*

Discussion stage	Discussion action	Discussion context
Evaluate	Instructor asks students to explain and evaluate the 'engineer' solution	*Debra, I appreciate you researching this problem and getting an engineer involved. I think he has a solution to be discussed further. Let's call it the 'engineer' solution and ask everybody to share their insights on this. Post your reaction on the 'engineer' solution, please. mt* *Here is my attempt to go against the engineer just to be difficult. The problem states 2.5 equals 2.4999… I think there is a difference of saying "exactly 2.5" and "infinitely close to 2.5". We can say that 2.4999… may have a limit but it will never be reached because it does on forever, so in reality there is a difference between both. Depending on your calculator 2.49! does not equal 2.5! If we consider this in a real word application and have two runners one a time of 2.49 sec and one with 2.5 sec who would be considered the winner? I think infinity is a concept and not a number, it's like saying 1/infinity = 0 you cannot divide a number by a concept. Jaime* *Hi Debbie, Thanks for posting the engineer's solution. I went from step to step, and realized it did make sense. I never had this mathematical training as most engineers would receive. A lot of my education, in my undergraduate work has been fully in the Liberal Arts category. It keeps reminding me of DNA how the match of 99.9999…% is essentially a complete or 100% match. It makes sense, after this supplemental solution. Again, it was very interesting viewing this! Brianna* *This question is really bothering me. My answer was A, because the question was very straightforward: "Which statement below is true?" And it is true that 2.49999… < 2.5. It does not matter how many 9's we add to the 2.499…. it will never reach 2.5, it will always be smaller than 2.5. I also have talked to some people, a PhD mathematics student told me that of course, 2.499 is smaller than 2.5, but that it will also depend on the context. Looking at the context of the question, my answer is still <. As an engineer myself, I know how critical is to work with decimals. Juan* *I actually enjoy reading the lively discussion this problem has created. I think it helped me see "proof" in a new way, and it was a good extension of our previous discussions. I believed the instructor also pushed us to come up with our own understanding of the challenging problem. Joanna*
Synthesize	Instructor brings a closure to the discussion	*Dear All, this was a thought provoking discussion and, most importantly, it exemplified the convincing a skeptic strategy that we have discussed last week. Let me synthesize the discussion.* *Juan made a good point that the solution to this problem "depends on the context." Pat earlier mentioned that "… mathematically, I think 2.4999…. is less than 2.5 because there is a very small difference in between these numbers." At the same time, Debbie presented the 'engineer' solution to the problem that convinced some of the participants: 2.4999…=2.5. Extending further, Jaime argued that "there is a difference of saying "exactly 2.5" and "infinitely close to 2.5."* *Thus, throughout the discussion we were looking at the same problem from the two distinctly different lenses: (1) the 'process' view (e.g., 2.4999… <2.5), and (2) the 'object' view (e.g., 2.4999… = 2.5). In mathematics education, this phenomenon is called 'process-object duality'. We will be further unpacking the idea of duality in our forthcoming discussions.* *Greatly appreciate everybody's input into this intellectually challenging yet engaging discussion. mt*

A well-designed and seamlessly implemented content interactivity and content communication significantly contribute to the effectiveness of learning environment in face-to-face and online education.

4.3. Engineering of Distance Learning

There are different Learning Management Systems used by universities across the globe for designing and offering online courses. In this section, we will describe the main features of the Blackboard Learn system to engineer distance learning using as an example one of the courses the author teaches at the University of Texas at El Paso — MTED5318 "Learning Theory in Mathematics Classroom" (Figure 34).

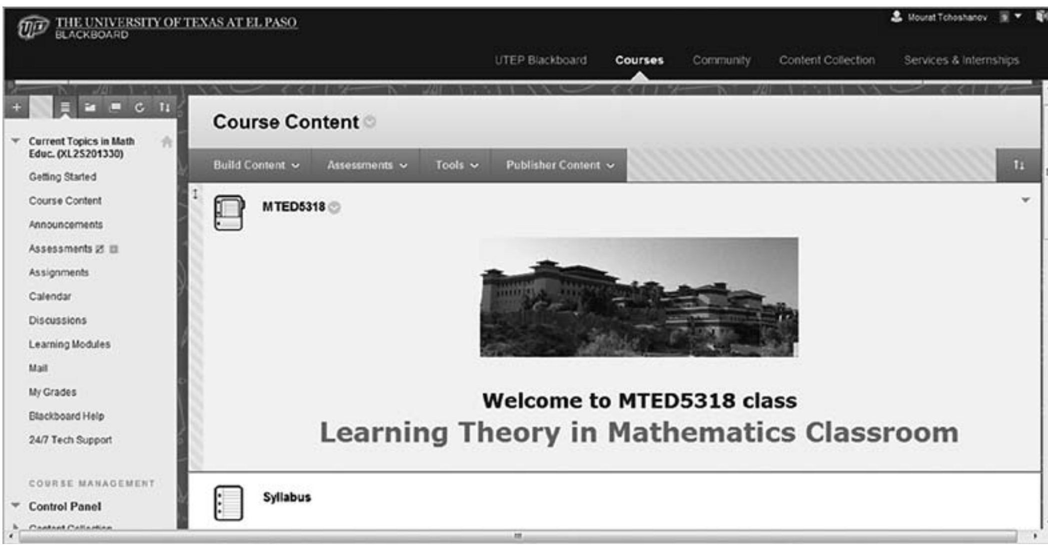

Fig. 34. Screenshot of the MTED5318 course homepage in the Blackboard learning system

The author has been using the learning management systems including Blackboard for designing and teaching courses in Mathematics and Mathematics methods for both pre-service and in-service teachers as well as graduate and doctoral courses in cultural-historical epistemology and learning sciences since 2005. As a designer and an instructor, the author is able to develop the course content, construct assignments, select ICT resources in connection with the course goals and objectives, design a system of monitoring and evaluating student progress, provide learning environment for student interaction (individually and in group) and communication using both synchronous (e.g., chat-room) and asynchronous modes (e.g., discussion board, blog, wiki).

The Blackboard Learn course environment consists of three main sets of tools: the Course Menu, the Control Panel, and the Course Content. The Course Menu central for the organization and navigation of the course is located on the left side of the course homepage. The designer uses the Course Menu tool to present the key links to the course materials such as "Getting Started", "Announcements", "Learning Modules", "Assignments", "Calendar"," Discussions", "Mail", etc. The appearance and the order of the links could be customized by the designer using two views available to users: (1) the list view, which displays only the top level of course materials, and (2) folder view, which displays the course materials as a directory tree.

The Control Panel is the course management tool, which consists of the following features: Content Collection, Course Tools, Evaluation, Grade Center, Users and Groups, Customization, Packages, and Help. The Content Collection is a repository of course files. Changes made to a file in this area will be automatically replicated to all of the courses where the file is used. The Course Tools includes communication, collaboration, assessments, and other tools available for use in the course. The Evaluation feature provides links to Course Reports, Performance Dashboard, and Early Warning to access the diagnostic information on student performance (e.g., activity and content usage). The Grade Center includes links to the assignments and assessments that need grading and the grade book/spreadsheet with students' grades. The Users and Groups feature consists of the list of students enrolled in the course and enables the designer to organize students into groups. The Customization helps the designer to manage properties such as course availability, tool availability, and course appearance. The Packages and Utilities are used to import, export, and archive the course. The "Help" feature provides access to the Blackboard Learn Guide and On Demand Learning Center.

The Course Content includes the following tools: Build Content, Assessments, Tools, and Publisher. The Build Content allows the designer develop the content of the course through uploading files, folders, posting syllabus, lesson plans, modules, and making links to external resources such as NBC Learn, for example. Assessments tool is used to create tests, surveys, assignments, etc. The Tools include discussion boards, blogs, wiki, etc. The "Publisher" provides links to the "Textbook publisher" and its additional resources such as "MyLab" and others.

All the tools used in the Blackboard Learn could be categorized into two main groups: (1) interactive tools and (2) evaluation tools. The Interactive Tools include announcements (e.g., notifying students about course events, assignment clarifications, and schedule changes), blogs (e.g., an online journal or a diary), calendar (e.g., important events and dates in the course), collaboration (a synchronous communication tool including a virtual classroom and a chatroom), contacts (instructor's contact information), discussion board (an asynchronous communication tool for creating forums), glossary (e.g., a list of important course terms and definitions), groups (e.g., creating and managing groups), journals (e.g., similar to a discussion board with a selective access to view journal entries), messages (e.g., similar to email), roster (the list of students enrolled in the class), email (sending email to students enrolled in the course), tasks (e.g., assigning as well as defining priority and tracking task status), wikis (allows students to collaborate on writing and editing course assignments).

The "Evaluation Tools" group consists of course reports (e.g., information about students' activity and content usage), an early warning system (sending email to students when the due date, test score, or other criteria and requirements are not met), full grade center (a grade book with main course assignments and tests), needs grading (e.g., items pending for grading), performance dashboard (an up-to-date report on students' activity and performance), pools (e.g., a repository for test, quiz, survey questions), rubrics (e.g., creating a qualitative assessment criteria), safe-assign/turn-it-in assignments (e.g., self-checking for plagiarism). The key for the evaluation tools is the grade center which resembles a spreadsheet and is designed as a dynamic and interactive tool allowing the instructor to record data, calculate grades, and monitor the student progress. It also permits to generate reports on student performance. The instructor can customize views and create grading schemas including grading periods, categories, and columns to present and gather the desired information.

While designing an online course, the author uses the *didactical engineering approach* which was discussed in Chapter 1. It includes three major stages: analysis, design, and construction of teaching products in order to create an effective learning environment for distance learning. Let us consider each stage in engineering of the MTED5318 course mentioned above.

Analysis. At this stage, the designer builds the foundation for the course through the study and analysis of standards, teacher competences, place and role of the course in the program, course description, texts and materials relevant to the topic of the course, digital media and ICT resources applicable to the course goals, teacher misconceptions, etc. The designer also selects the required textbook and readings materials as well as defines learning objectives at the analysis stage.

The MTED 5318 is a semester-long graduate level topic course. It is aimed to "develop competencies necessary to deal effectively with Mathematics instruction; includes curriculum, concepts, teaching strategies, and skills necessary to integrate content and teaching strategies". The author selected the topic of "Learning theory in Mathematics classroom" for this course and defined the following learning objectives. Upon completion of this course students should be able to:

1. know and understand the guiding principles of learning Mathematics in the classroom;
2. apply the guiding principles in the Mathematics classroom in order to develop students' conceptual understanding and procedural fluency;
3. analyze and reflect upon implementation of the guiding principles in the middle school Mathematics classroom;
4. evaluate effectiveness of learning in the Mathematics classroom using selected ICT resources.

Through the extensive and careful analysis of available resources and materials relevant to the topic and learning objectives of the course, the author identified the following required texts for the course:

- National Research Council (2005). How Students Learn Mathematics in the Classroom. M. Suzanne Donovan & John D. Bransford, Eds. Washington, DC: The National Academies Press (available online at http://books.nap.edu/catalog/10126.html)
- Boaler, J., & Humphreys, C. (2005). Connecting Mathematical Ideas: Middle school video cases to support teaching and learning. Portsmouth, NH: Heinemann.

The design stage focuses on the development of student learning pathway based on the learning objectives defined at the analysis stage. At the design stage, the instructor is primarily concerned with the connection between the learning objectives, the course content, and the assessment of learning outcomes. It order to achieve the seamless connection, the instructor carefully designs and selects the major course activities, assignments and course deliverables. The assessment of learning outcomes in connection with the learning activities and assessment is shown in Table 9.

Table 9. Connecting learning objectives, learning activities, and learning outcomes

Learning Outcome	Achieved by	Measured by
To know and understand the guiding principles of learning Mathematics in the classroom	Readings and reflections Participation in discussions	Concept Test Participation Checklist
To apply the guiding principles in the Mathematics classroom in order to develop students' conceptual understanding and procedural fluency	Applying activities in the classroom Participation in discussions	Lesson Plan and Video of Lesson Participation Checklist
To analyze and reflect upon implementation of the guiding principles in the middle school Mathematics classroom	Reflections on video cases Participation in discussions	Written Reflection Participation Checklist
To evaluate the effectiveness of learning in Mathematics classroom using selected ICT resources	Reflections on ICT enhanced activities Participation in discussions	Written Reflection Participation Checklist

The construction stage builds on the design stage and aims at the selection and development of particular teaching products including but not limited to syllabus, modules, lessons, assignments, ICT resources, etc. It also aims at creating an effective online learning environment.

The syllabus is the key document defining the course objectives, its content, requirements, and assessment. Structurally the syllabus may include the following components: title and description of the course; contact information about the developer/instructor of the course, including virtual office hours; learning objectives; connection between the objectives of the course, its content and assessment; textbooks and reading materials used in the course; schedule of classes and activities; list of course assignments; grading scale, rubrics, class participation requirements; software requirements; professionalism and academic integrity statements.

The course content for the MTED5318 includes eight problem solving activities aligned with reflections on eight video cases of the middle school Mathematics classroom, five chapters from the required text on "How Student Learn" with corresponding chapter tests, analysis, and reflection on five selected ICT resources, four lesson plans for selected activities with classroom teaching and videotaping, a collection of student work, and participation in class discussions. There is a mixture of individual and group activities in the course. Problem solving as well as lesson plan design, teaching, and classroom videotaping are group activities whereas reflections, chapter tests, and participation in discussions are individual activities.

At the construction stage, the designer uploads the course content into the LMS and selects the main interactive and evaluations tools to support and monitor student learning over the course duration. The author usually uploads the course materials two weeks prior to the beginning of the class, so the students have an access to view the syllabus and major requirements of the course. The students may also need an extra time to order the required texts for the course.

The "Getting Started" feature at the course menu is an important step in the beginning of the course where the instructor introduces himself/herself and sends a welcome message and an invitation to the course. The "getting started" message may also outline the key information for students to help them get off to a good start such as friendly suggestions, communication and software requirements, support system, etc. (Figure 35).

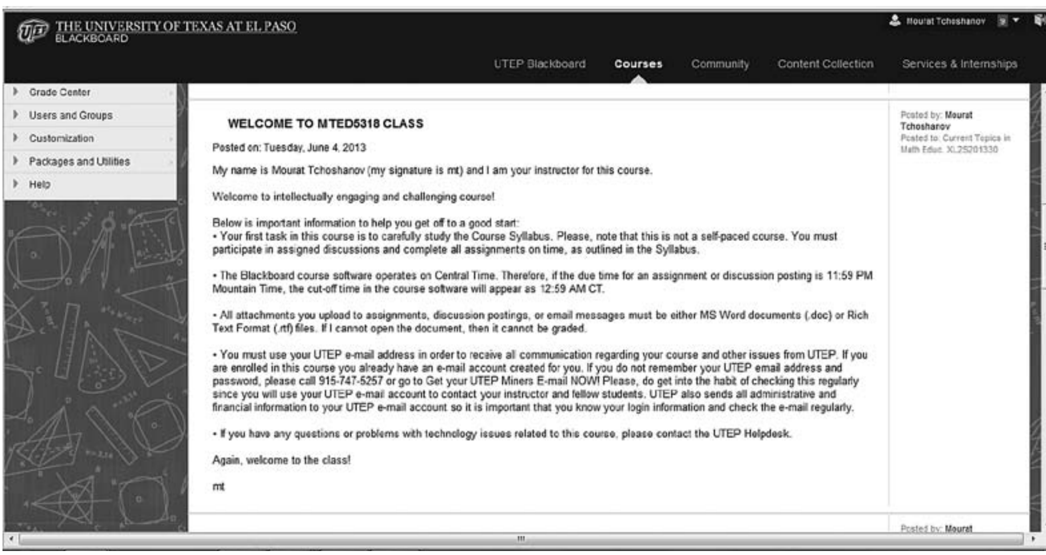

Fig. 35. Screenshot of the "Getting Started" message for the MTED5318 course

One of the central course assignments is problem solving, analysis, reflection, lesson planning, and teaching based on the selected video cases of the middle school mathematics classroom. This assignment is a connected set of activities that addresses the learning objectives of the course and outlines the student learning pathway. It consists of the following steps:

- pre-video problem solving activity;
- during-video analysis of the didactical situations occurring in the video;
- post-video reflection;
- post-video discussion;
- video-based lesson plan development;
- pre-teaching conference;
- teaching and videotaping the lesson;
- reviewing and analyzing the classroom video.

The first step is to involve students in solving the problem that later they will watch in the video. Let us consider the "Border problem" from the first video episode as an example. The instructor divides the class into small groups to work on the following activity:

Problem 1: Using the 10 by 10 grid (Figure 36), figure out *without writing and without counting one by one*, how many unit squares are in the colored border of the grid? Explain your method.

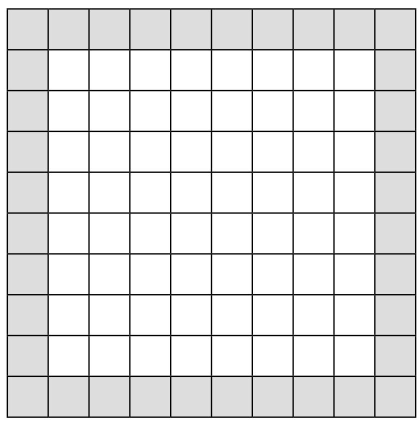

Fig. 36. Border problem (version 1)

Problem 2: How many unit squares are in the colored border of the grid below (Figure 37)? Explain your method.

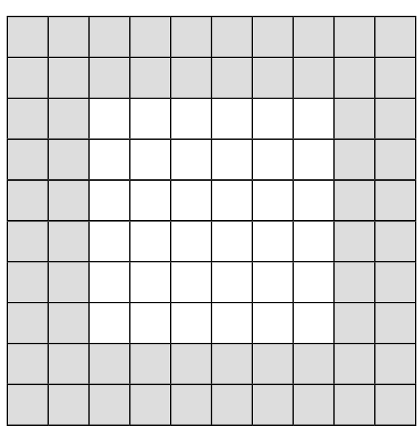

Fig. 37. Border problem (version 2)

The groups post their solutions and explanations on the discussion board. The whole class has a chance to comment on the posted group solutions before the next online session where the instructor provides an access to the video case (Figure 38).

Fig. 38. Video case based problem solving activity

"During-video" activity includes an element of *gamification*. The game-based learning activity designed for this assignment is called "*Didactical Chess*". The *o*bjective of the game is to watch the video case until the pause point purposefully selected by the instructor, analyze the situation, come up with the most effective didactical move, and justify why the selected move is the most effective with regard to student learning.

Below are the steps of the game. First, the designer/instructor carefully selects the didactical situation(s) in the video that include teachable moments, such as student ways of problem solving, student misconceptions, student questions, etc. The designer includes the pause point in the video track. The students watch the video until that very point and during the pause they individually work on the "Engineering of a didactical move" chart which includes the following segments.

1. Analysing the situation:
    - analyse the content
    - analyse the teacher action(s)
    - analyse the student action(s)
    - analyse the classroom environment
2. Designing possible didactical moves
    - define the didactical task
    - identify the main factors impacting the potential move(s)
    - list possible didactical moves

3. Selecting the move and constructing the next didactical situation
    - select the most effective didactical move
    - justify why it is effective
    - construct a similar didactical situation.

The main purpose is to come up with the next most effective move, if they were teachers in the video. Students can rewind the video back but they cannot continue watching the video until they submit the response. The most intriguing part for the students after completing the response is to watch the teacher action in the video case. It usually happens that few students might come up exactly with the same move as in the video. The designer/instructor usually selects one didactical situation per video case. The "Didactical Chess" activity involves teachers in zooming into details and dissecting the "molecular" didactical situations into "teaching atoms". Good teaching is about being able to conduct microscopic analysis of teaching craft and, based on this analysis, to understand how to effectively engineer student learning. After all, 'the devil is in the details'. The post-video activity includes an individual student reflection on the entire video case. The sequence in the video lesson included the following events (adapted from Boaler & Humphreys, 2005): the teacher collected "wrong" answers to the "Border Problem" and asked students' reaction and thinking about the incorrect answers; the teacher collected different methods for thinking about the correct solution; the teacher gave a method from the previous day's class and asked students to make sense of it geometrically; the teacher initiated the discussion of the similarities and differences between methods; the teacher posed a question about shrinking the square to a 6-by-6 grid and there was some discussion of the proposed student answers.

The reflection was supported by the following guiding questions divided into four main areas: (1) the activity, (2) the teacher; (3) the students; (4) the classroom environment (Boaler & Humphreys, 2005).

The Activity section includes the following questions for reflection: What were the mathematical tasks of the lesson? How did they follow from the main activity? What do you think about each of the events in the lesson? What do you think about the progression of the events? What Mathematics means did each of them suggest? What were the decision points in the lesson that had changed the flow of the activity and when did they occur? Were there any didactical situations in the lesson you would have approached differently? What mathematical content and mathematical process did the lesson address? Where could this lesson go from here? What could students work on during in the next lesson?

The Teacher section of the reflection consists of the following questions: How did the teacher respond to student's different methods? How did the teacher capitalize upon student's diverse way of thinking? How did the teacher gather information from the students? What kinds of information did s/he gather? What would you have done differently if you were the teacher? At which didactical situations would you have made different decisions and why?

The Students section includes the following questions: What did students learn in this lesson? Do you think it was different for different students? How? Why? What were the various roles students played in the classroom? What different things were the students required doing? What questions did students ask? Which students were contributing or not contributing to the discussion?

Finally, *the Environment* area of the reflection includes the following set of questions: What classroom norms did you see in this class? What do you think the teacher had done to set up these norms? How was the classroom arranged? What materials were used and which role did they play? What in the classroom environment made the Mathematics more visible?

After the students submit individual reflections, the instructor invites them to the post-video discussion. An example of the invitation to the discussion on video cases consisting of the "Border Problem" is presented below.

"Dear All,

It took me a little longer to read first two reflections. At the same time, I have had enough time to think how to respond to the issues that were challenging to the most of you. Based on your reflections, I feel that many of you enjoyed watching Cathy's teaching. I did too. Particularly, I value her way of ENGAGING students in learning and understanding of challenging topics in Algebra such as pattern generalization, concept of variable and notion of proof. The first comment I would like to make is don't shy away from digging deeper in the content. After all, this is a class on learning MATHEMATICS.

Now, let me share my observations on some of the important content-specific issues related to video cases 1 and 2. The ideas of a pattern generalization (video case 1) and a variable (video case 2) are key concepts. In video case 2, Cathy nicely pushes students to use letters in order to come with an algebraic expression for Joe's case. I wish Cathy would make a very important distinction in the role letters play in algebraic representations. Letters could play a dual role: for example, in an equation $4x — 4 = 36$, x is unknown, in a function $y = 4x — 4$, x is variable. The case when x is variable takes care of students' confusion on the question "What is staying the same and what is changing?" "4" is staying the same as a number of sides for any square as well as "-4" stays the same because you always have to take off four overlapping unit squares in the corners, and x is changing (it could be 10 for the case of 10x10 square, it could be 6 for the case of 6x6 square, it could be 100 for the case of 100x100, and so on).

Understanding what is a variable, also addresses the question "How might you explain to students when another variable is needed?" In one of the episodes, Pam suggested to use two variables s and n=s-2 for the side lengths. How would you address this issue?

At the same time, in the expression $y = 4x — 4$ we, indeed, have two variables: an independent variable x — the number of unit squares on one side of a given square, and a dependent variable y — the total number of unit squares on the border of *a given square. So, if x=10, then y=36. If x=6, then y=20, and so on. If so, do we need two independent variables in this case!? Does this make sense to you? If your answer is"yes", explain WHY? If "not", share your concerns, please. mt"*

Through the invitation, the instructor encourages the students to dig deeper into the important content-specific issues addressed in the video case. Unfortunately, some of the middle school teachers have a tendency to use a general and descriptive way of reflecting on video cases. Some of them shy away from the content. Instructor's role is to engage the students into the content-focused discourse and sustain the focus throughout the discussion. An example of the extended description of the discussion to promote content communication was presented in Section 4.2 (Table 8). After the individual participation in the discussion, the students work in groups on developing the lesson plan based on the same video case. When the draft of the lesson plan is ready, the group submits it to the instructor and requests a virtual office hour to conduct a pre-teaching conference. The instructor holds a synchronous conference with the group via chat room or Skype and provides a feedback on the lesson plan developed by the group. The second synchronous post-teaching conference is conducted after the group has taught the lesson and submitted the videotape to the instructor. The post-teaching conference concludes the engineering of teacher learning cycle: teacher learning — lesson planning — teaching practice — student learning (Figure 39).

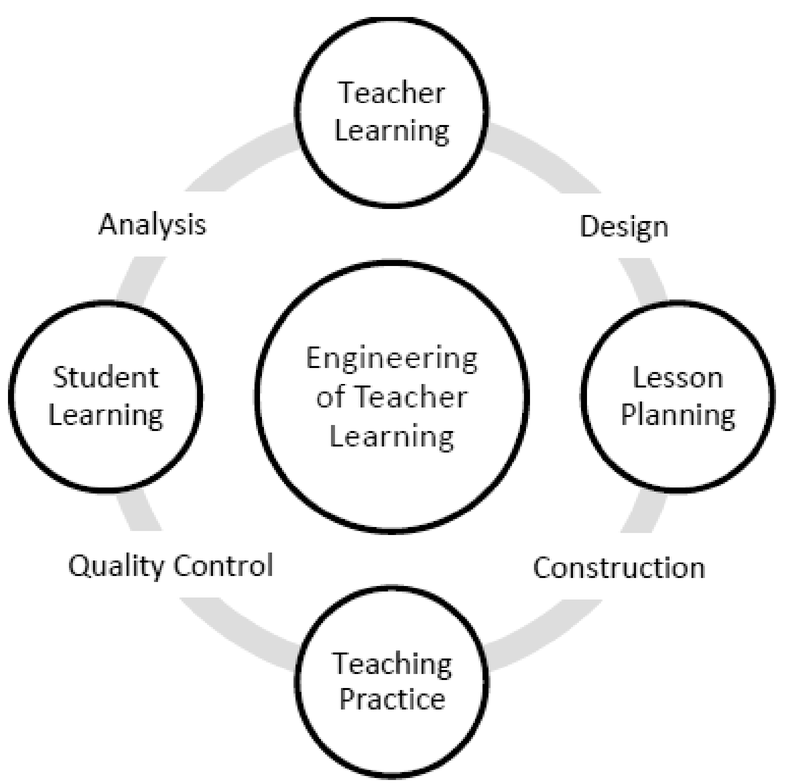

Fig. 39. Engineering of teacher learning cycle

Another activity used in the online MTED5318 class is an analysis and reflection on selected digital interactive ICT resources. For example, in order to achieve the course objective — "To evaluate effectiveness of learning in Mathematics classroom using selected ICT resources" — students are assigned the task: "Explore the e-example", "Understanding Distance, Speed, and Time Relationships Using Simulation Software" (Figure 40: http://standards.nctm.org/document/eexamples/chap5/5.2/index.htm). Set up several trials using the simulation applet. You may try out this activity in your own classroom. Evaluate the effectiveness of this activity in promoting student learning. Reflect on the following question: what big ideas about functions and representing change over time students learn while working on this activity?"

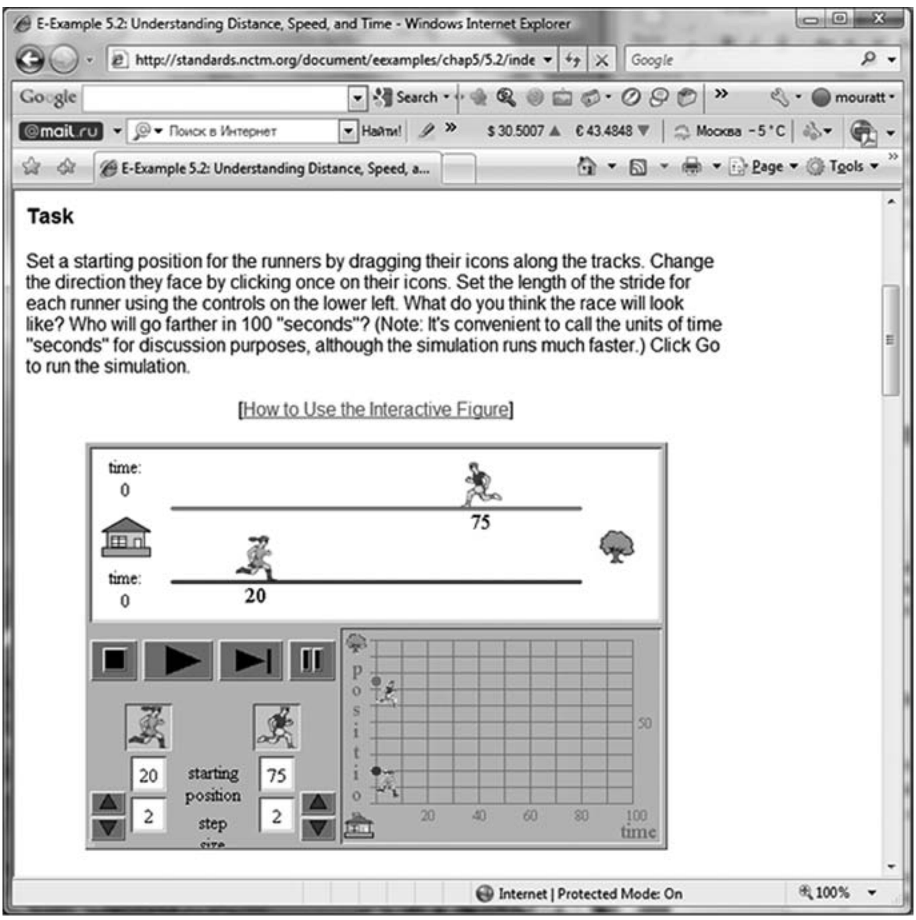

Fig. 40. Screenshot of a digital interactive learning resource

Throughout the course activities, the instructor continuously encourages students to focus on the content and develop their pedagogical knowledge and skills. Another way to emphasize the importance of content-specificity is to construct and provide rubrics for students' reflection and participation in the discussion. An example of such a rubric is presented in Table 10.

Table 10. An example of the rubrics for assessment of students' reflections and discussions

Quality	High 4.0 Points	Good 3.0 Points	Satisfactory 2.0 Points	Poor 1.0 Point
Completeness	Responds to all questions	Responds to most questions	Responds to few questions	Responds to one question or does not respond a single question
Clarity and Details	The main idea stands out and is supported by detailed and content-specific information	The main idea is clear but the supporting information is too general	The main idea is somewhat clear but there is a need for more supporting information	The main idea is not clear. There is a seemingly random collection of information
Accuracy	All supportive facts are reported accurately	Almost all supportive facts are reported accurately	Few supportive facts are reported accurately	No facts are reported or most are inaccurately reported
Resources	All resources used for quotes and facts are credible and cited correctly using APA format	Most resources used for quotes and facts are credible and cited correctly using APA format	Few resources used for quotes and facts are credible and cited incorrectly	The resources used for quotes and facts are less than credible (suspect) and cited incorrectly
Grammar	No errors in grammar or spelling distracting reader's attention	One or two errors in grammar or spelling that distract the reader from the content	Three of four errors in grammar or spelling that distract the reader from the content	More than 4 errors in grammar or spelling that distract reader's attention from the content

At the end of the semester, the students develop an e-folio, which includes all the major assignments for the course including problem solving, reflections, chapter tests, lesson plans, classroom videos as well as students' contribution to the class discussions.

The students' course evaluations conducted at the end of the semester reflect the level of intellectual challenge, as well as benefits the course offers for both pre- and in-service teachers enrolled in the class. Below are the samples of students' evaluations.

"This course has been a great experience for me. It was challenging and thought provoking. The assignments have been challenging too and have addressed mathematical topics that will contribute to enhancing my teaching perspective and experience. The course was well facilitated. The teacher provided ample time and instruction for the assignments to be understood and completed. I can definitely say that I have enjoyed being the part of this online course."

"This course was extremely rigorous! It is absolutely the best course I have taken at the UTEP[7] and should be a mandatory course for anyone pursuing a degree in teaching. I had learned far more than I expected to learn, and left with many ideas for things I could do to help my future students understand math. Other courses I have had at UTEP

[7] University of Texas at El Paso

have been very theoretical or have asked me (someone with very limited experiences in the classroom) to come up with lesson plans or discipline plans (which I really didn't know anything about — so the lesson plans I came up with were not based in reality). This course GAVE me the lesson ideas, showed me the ways of teaching, and then asked me to evaluate which ones I thought would be most effective or least effective. Instead of having me create lessons out of thin air with no past experience to build on, I had the chance to observe great lessons — I now have some ideas to emulate. I LOVE this model. It's amazingly helpful. I will gladly take any other courses Dr. T is teaching (but I'll be sure I'm not too busy because this class was TOUGH and took much more time than I expected). However I learned a great deal for my investment in time and money, and I really feel that this course was a good investment."

"I really do want to give high remarks for this professor for the selection in course textbooks and assignments. Out of all the ATCP[8] *courses I have taken, I feel this has been the most beneficial, effective, and the most that I have learned. The professor was very good in responding and giving good discussion questions and his own discussion remarks. Despite being an online course, the instructor was very efficient in answering e-mails and assisting students with technical problems."*

Synthesizing the multiyear engagement in designing and teaching online courses, the author concludes that it is a challenging yet a rewarding experience that requires seamless integration of content, didactical and engineering knowledge, and skills to create an effective learning environment in distance education. This new type of knowledge the author calls CODE = *content-oriented didactical engineering* knowledge. The author believes that this type of knowledge emerges as a critically important code to unlock challenges of designing effective learning environments and improving teacher competence in the digital age.

[8] Alternative Teacher Certification Program, which is a part of the Teacher Education Program at the University of Texas at El Paso.

Conclusion

In today's world, current revolutionary changes are associated with the intensive use of digital technologies in many spheres of human life, which democratize knowledge and access to open education. The ICT is increasingly implemented in the daily lives of individuals and the society. We are witnessing the formation of a new phenomenon — a global virtual learning community, which today includes more than one billion users. And the numbers continue to grow. Along with this, the market of online educational services is steadily growing. For example, in the Department of Teacher Education at the University of Texas at El Paso (USA) about 50% of graduate courses are conducted in an online format. With the purpose of expanding online services, the leading universities create MOOC consortiums (e.g., Coursera, Udacity, edX, etc.) to initiate special programs for supporting the design and delivery of online courses, as well as the development of new tools for online learning systems. This creates a domino effect: along with the transfer of many university disciplines, including teacher education courses to the online format, there is a need to revisit the training of school teachers. Instead of the traditional teacher training, the focus is shifting toward a new type of training for teachers who can work in the digital age, with high demands on teachers' knowledge and ability to engineer an effective online learning. Moreover, in the digital era a teacher is not just an online tutor, s/he becomes an analyst and manager of informational resources, a designer and a constructor of courses, modules, and lesson fragments using interactive multimedia tools.

In connection with the emerging changes in the role of teachers in the digital age an important question arises: what kind of teacher is needed in the digital age? In order to meet the demands of the new era, a teacher in a traditional sense (e.g., someone who teaches) should be replaced by a teacher-engineer (e.g., someone who engineers student learning). This shift comprises integration of teacher knowledge of content, engineering, and didactics. At the same time, the integration implies reconceptualization of the key role of *a teacher-engineer* in the digital age: traditional teaching transforms into *a research-based engineering of student learning*. This transformation requires a teacher-engineer to understand the teaching theory and learning sciences in order to effectively design the learning objectives, digital content, and assessment, and to connect them.

The 'engineering of learning' paradigm places a critical emphasis on the development of teachers' engineering design thinking. The development of teacher-engineer's design thinking is a complex process based on the advancements of the learning sciences. It involves the following key competences:

1) the design of learning objectives: to create outcome-based, technology-enhanced learning environments that enable students to set their own learning objectives, monitor and assess their learning progress;

2) the engineering of content: to develop interactive content and relevant learning experiences through the selection and design of tasks, problems, projects, and activities that incorporate digital tools and ICT resources to promote student learning and creativity;
3) the design of assessment: to select and develop authentic assessments aligned with the learning objectives and content, and to use assessment data to improve teaching and promote student learning.

In order to respond to the challenges of the digital age, didactics itself needs to be re-conceptualized. This re-conceptualization has a clearly defined vector. Modern didactics is moving towards strengthening its “engineering” functions — didactical engineering. We call this trend *e-Didactics* and define it as ICT-integrated didactics with its major focus on *engineering of learning.*

Didactical engineering is a relatively new approach in education. It focuses on the “precise” design of the learning process, which can later be reproduced in other “point” of time and space under the predetermined conditions.

e-Didactics aims to use scientific methods and promotes the formation of teachers’ design thinking. e-Didactics also fosters the development of teachers’ analytic reasoning focused on the implementation of macro and micro analysis of didactical systems, processes and situations. Accordingly, e-Didactics has its own subject domain that is characterized by the following main parameters: analysis, design and construction of outcome-oriented teaching products (e.g., learning technologies); application of a scientific method and design thinking into the analysis of didactical systems, processes, and situations in order to create effective learning environments.

The development of didactics in the direction of the e-Didactics and didactical engineering offers new opportunities for further understanding of learning and teaching in the digital age and creating effective learning environments in an emerging global learning community.

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WEBSITES

Screenshot of the Cognitive Tutor Algebra system —
http://www.carnegielearning.com/galleries/4/

Open source for the e-portfolio development —
http://www.mahara.org

Author's open access website on Visual Mathematics —
http://mourat.utep.edu/vis_math/

Open source Wolfram Demonstrations Project —
http://demonstrations.wolfram.com/AreaUnderACycloidII/

Open source Scientific Visualization project —
http://upload.wikimedia.org/wikipedia/commons/a/a2/Tesseract.ogv

Open access Mathematics and Science Education website —
http://www.svsu.edu/mathsci-center/uploads/math/gmconcept.htm

NBC Learn media streaming resource —
http://www.nbclearn.com/portal/site/learn/

Khan Academy — https://www.khanacademy.org/math/trigonometry/functions_and_graphs/undefined_indeterminate/v/undefined-and-indeterminate

Open access Function Game — http://www.functiongame.com/

Open access NCTM e-examples —
http://standards.nctm.org/document/eexamples/chap5/5.2/index.htm

UNIT[9] "LEARNING PATHWAY TO PROPORTIONAL REASONING" LEARNING OBJECTIVES

After successful completion of this unit, the student is expected to:

- apply mathematical process standards to use proportional and non-proportional relationships to develop foundational concepts of functions;
- represent linear proportional situations with tables, graphs, and equations in the form of $y = kx$;
- represent linear non-proportional situations with tables, graphs, and equations in the form of $y = mx + b$, where $b \neq 0$;
- contrast the bivariate sets of data that suggest a linear relationship with the bivariate sets of data that do not suggest a linear relationship from a graphical representation;
- use a trend line that approximates the linear relationship between bivariate sets of data to make predictions;
- solve problems involving direct variation;
- distinguish between proportional and non-proportional situations using tables, graphs, and equations in the form $y = kx$ or $y = mx + b$, where $b \neq 0$.

[9] The unit was developed by the Texas-Science, Technology, Engineering, and Mathematics (T-STEM) team: A. Duval, D. Garcia, L. Michal, M. Tchoshanov, and A. Torres. This is a reduced version of the unit.

UNIT CONTENT

Pre-assessment (not included)

Introductory Elements

- Unit Objectives
- Prior Knowledge
- Unit Map
- Unit Project
- Language and Communication

Unit Core

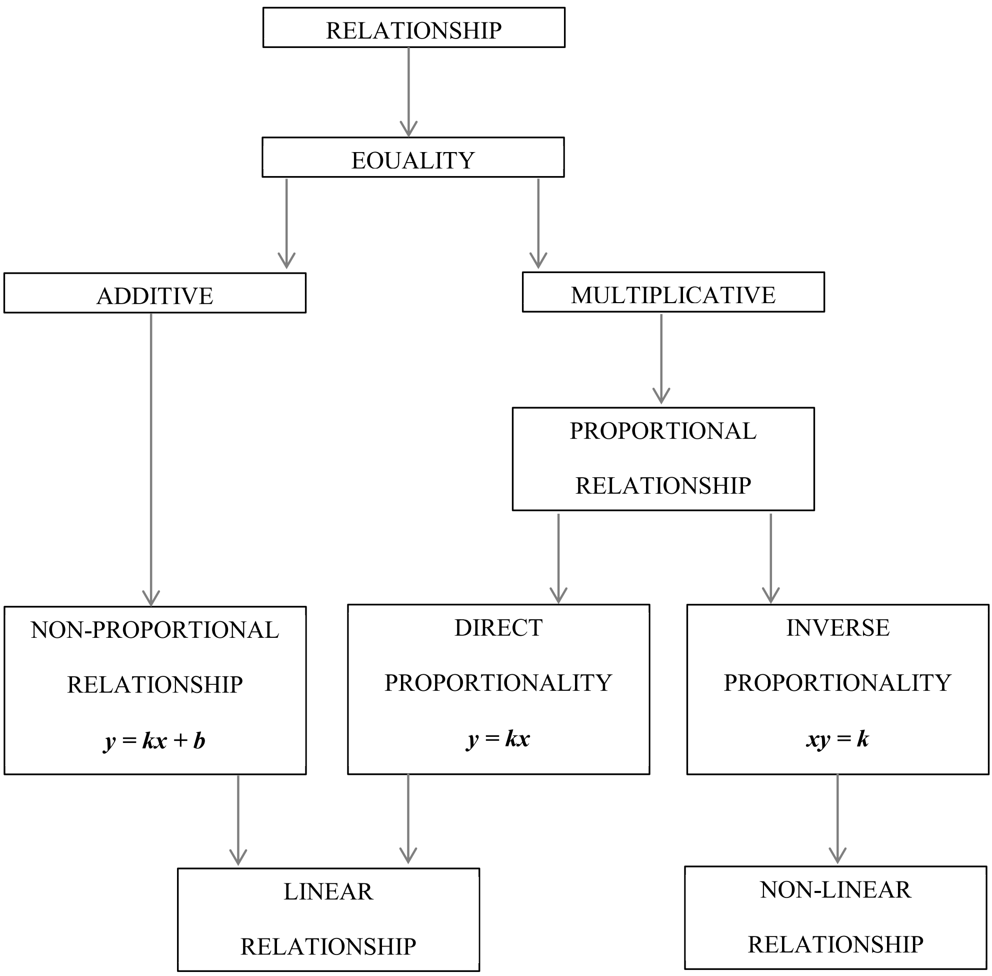
- Mission One
- Mission Two
- Mission Three
- Mission Four

Applied Elements

- Applications
- Connections
- Generalization
- Debugging
- Extensions

Post-assessment (not included)

UNIT MAP


RELATIONSHIP
EOUALITY
ADDITIVE
MULTIPLICATIVE
PROPORTIONAL RELATIONSHIP
NON-PROPORTIONAL RELATIONSHIP
$y = kx + b$
DIRECT PROPORTIONALITY
$y = kx$
INVERSE PROPORTIONALITY
$xy = k$
LINEAR RELATIONSHIP
NON-LINEAR RELATIONSHIP

PRIOR KNOWLEDGE: IDEA OF RELATIONSHIP

Dollars and Pesos

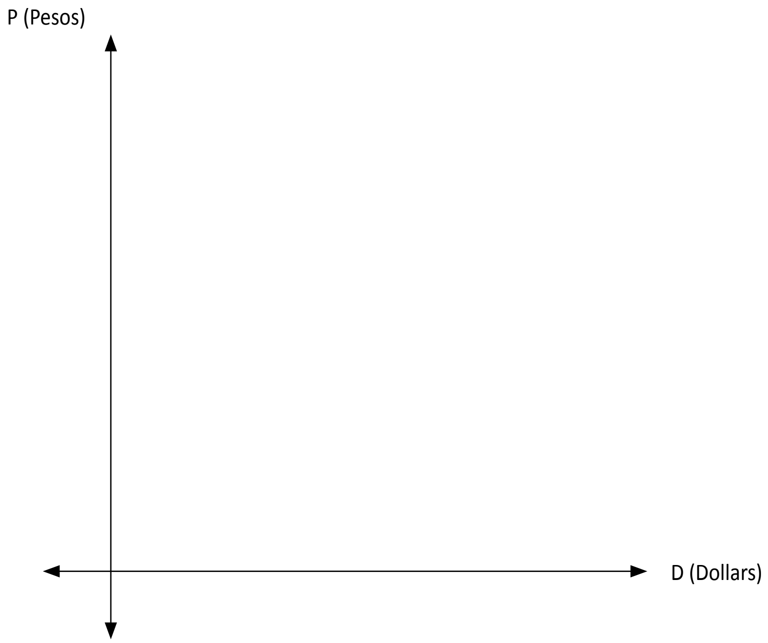
1. What did the newspaper state as the exchange rate between dollars and pesos?
2. On July 14, 2007, the exchange rate from U. S. dollars to Mexican pesos was stated as: 1 US dollar = 10.77 Mexican pesos.

   Round* 10.77 to the nearest peso: 1 US dollar = ____ Mexican pesos
3. Place one dollar on the table and the number of pesos we will use from 2 above to remind us of the exchange rate.
4. Write this in the ratio form*.
5. What other ratio form could we use?
6. Using your work with proportions*, set up a proportion with this ratio, use D for US dollars and P for Mexican pesos.
7. Write a statement in words you would use to tell someone how you convert dollars to pesos.

   Write a statement in words that you would use to tell someone how to convert pesos to dollars.
8. Fill in the table of values* for dollars or pesos. Write the mathematics you are using in the process column (D = U. S. Dollars and P = Mexican pesos).

Dollars	Process	Pesos	(D, P)
0			
1			
2			
3			
4			
5			
		200	
		300	
		400	
		500	

Graph the table of values on a graph with dollars on the horizontal axis* and pesos on the vertical axis*.



9. In the table of values and the graph of the pairs of numbers we get, what is happening each time we have the number of dollars and want to find the number of pesos?

   Here, let D = x and P = y and we have $y = x \cdot 11$ *or* $y = 11 \cdot x$

So the equation, the table, and the graph represent the same relationship. We are multiplying all the numbers along the "dollars axis", that is, the horizontal real number line*, to get the new numbers that we will be using for "pesos axis", that is the vertical real number line*.

* Prior knowledge students may bring into the classroom from their experiences but also to assess the prior knowledge that has been formally taught in previous mathematics classes denoted with an asterisk.

UNIT PROJECT: AN IDEA OF MULTIPLICATIVE RELATIONSHIP

Balancing Activity

For this activity you will need a balance and some blocks of different weights. You will be placing these blocks weights at different distances from the center, or the fulcrum of the balance.

Identical Weights

1. Get two blocks of the same weight. Put one block 3 cm to the left of the fulcrum. Where on the right side of the fulcrum do you need to place the other block so that it balances the first block?
2. Next, move the block on the left 1 cm further away from the center. Where do you need to place the block on the right side so that the balance stays balanced?
3. Now move the left block 5 cm away from the center. Where do you need to place the block on the right side so that the balance stays balanced?
4. What do you observe about where the blocks have to be placed to keep balance?

   Try some more experiments. Write down, in your own words, what you have observed.

Different Weights

Now you are going to balance a different number of blocks on the left and right sides of the fulcrum. Make two groups of blocks, where one group (Group L) will weigh double the other group (Group R). Group L will be placed on the left side of the fulcrum and Group R will always be placed on the right side of the fulcrum.

My Group L has ________ blocks and Group R has ________ blocks.

4. Place Group L 3 cm to the left of the center of the balance. Place Group R 3 cm to the right of the center. Describe what is happening?
6. Where do you need to place Group R to balance two sides?
7. Now move Group L 1 cm further to the left. Where do you need to place Group R so that the balance stays balanced?
8. How does this compare to the previous situation with the identical weights?

Making a table

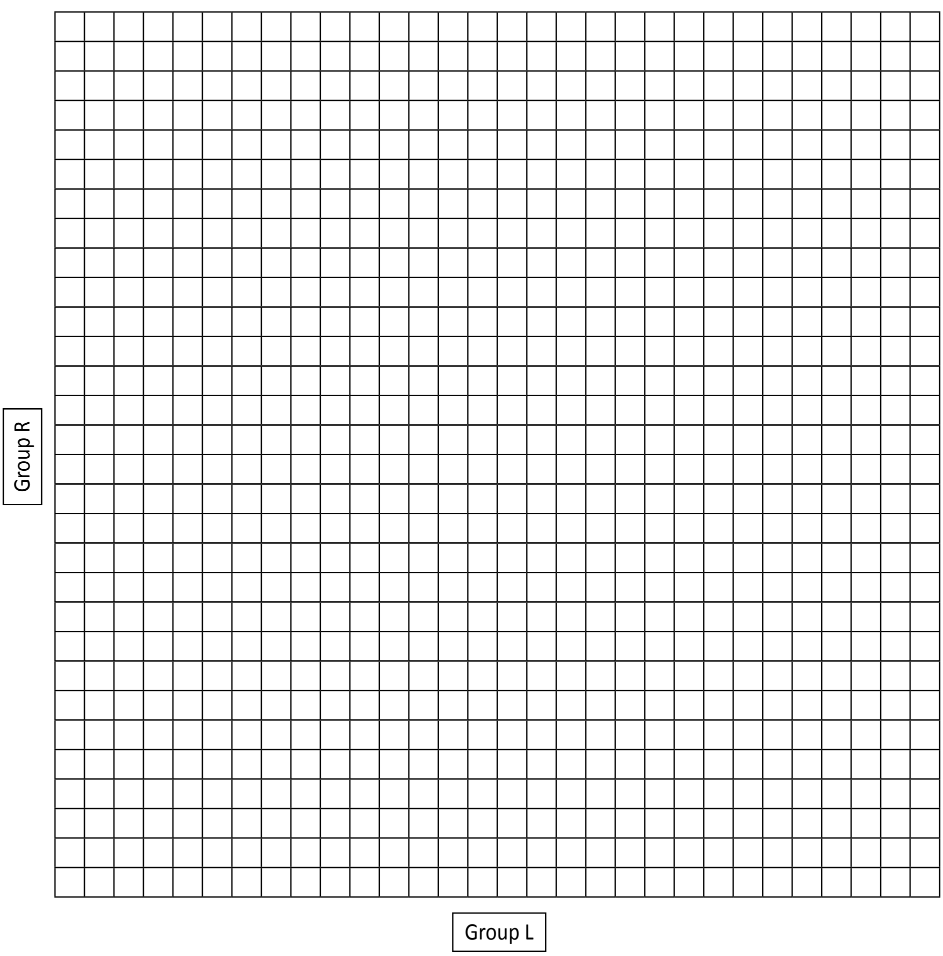
Complete the table that compares the distances from the fulcrum for Group L and Group R.

Group L	Group R
1 cm	
2 cm	
3 cm	
4 cm	
5 cm	
6 cm	
7 cm	

9. How much further out does Group R move every time you move Group L 1 cm?

Making a graph

10. Graph the data from the table.



11. What do you notice about the points you plotted?

Making predictions

12. If you had a bigger scale and could put Group L 8 cm from the center, where do you think you would have to put Group R? Explain your answer using both your table and your graph.

13. Suppose you could put Group L 10 cm from the center. Where do you think you would have to put Group R? Again, try to use both your table and your graph to explain your answer.

Historical Note

Archimedes, a Greek mathematician, was the first to explain the principle of the lever. Although he did not prove this principle, he was the first to state, *weights of equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance.*

When weights are equal, distances of the weights from the fulcrum must be adjusted to have a balanced state of equilibrium. The Archimedes' lever principle tells us,

if $W_1D_1 = W_2D_2$, then the above is in static equilibrium, with all torques balanced.

The distance from the point where you place the weight W_1 to the fulcrum is the lever arm distance $D_{1,}$ the distance from the point where you place the weight W_2 to the fulcrum is the lever arm distance D_2. Distances are measured from the fulcrum to the weights. Archimedes is said to have stated, "give me a place to stand on, and I will move the Earth". Imagine you have a huge lever on one side of which you have the Earth and on the other side a place to stand. How far would you have to stand to move the Earth?

Mathematical Note

From Physics, torque, force, and weight are given by the following.

$T = F \cdot D$ Torque is force times lever arm distance.

$F = M \cdot g$ Force is mass times acceleration.

$W = M \cdot g$ Weight is mass times acceleration of gravity on earth.

To have a balanced state, Torque_1 must equal Torque_2, that is,

$T_1 = T_2$.

$F_1 \cdot D_1 = F_2 \cdot D_2$ Substitute force times distance for torque.

$M_1 \cdot g \cdot D_1 = M_2 \cdot g \cdot D_2$ Substitute mass times acceleration for force.

$W_1 \cdot D_1 = W_2 \cdot D_2$ Substitute weight for mass times acceleration.

ATLANTIS MISSION ONE

Representing Proportional Relationships

Along with astronauts, the International Space Station, ISS, has also hosted tourists from planet Earth. Between 2001 and 2007, five tourists have traveled to the ISS at an average cost of $25 million per person. Currently, 200 seats for tourists have been presold.

1. Complete the table below to show the cost of different numbers of tourists if the cost remains the same. Let t represent the number of tourists and c represent the cost of tours in millions.

Number of tourists (t)	0	1	2	3	4	5	6	7	8	9	10	...	t
Cost of tourists (c)													

2. Write how you would find the cost of 11 tours.

3. Explain the rule you would use to find the cost of any number of tourists (t).

4. Graph the relationship between the number of tourists and the cost of tours on the coordinate grid provided.

5. Use your rule to write an equation that describes the relationship between the number of tourists, t, and the cost of trip, c.

Mathematical Note

Relationship is equality, inequality, or any property for two objects in a specified order, for example $a=b$, $a<b$, $ab=ba$, etc.

Ratio is a multiplicative comparison between two quantities. Rate is a reflectively abstracted constant ratio. *Unit rate* is a rate with the second term equal to one.

A relationship between two varying quantities is **proportional** when the ratio of corresponding terms is constant.

6. What is the rate of tourists to cost? ____________________

   What is the rate of cost to tourists? ____________________

   What is the unit rate of cost to tourists? ____________________

7. What will the cost be for 200 tourists? ____________________

   Use a proportion and the equation to verify this cost.

Using a Proportion	Using an Equation

8. How many tourists would be able to travel for $425 million?

   Show your work.

9. Identify where the unit rate appears in the table, the graph, the equation, and the proportion.

____

____

____

____

ATLANTIS MISSION TWO
Proportional Relationships

The International Space Station, ISS, relies on solar energy to power itself. It needs a total of eight solar array panels. The ISS used the Mercury Solar Company to buy the commercial size solar panels. The cost of each panel is $19 million. Due to the structural constraints of the space station, the panels must be installed in pairs.

1. Write a ratio that compares the number of panels to the cost of panels.
2. What would the cost of three panels be? Use this information to determine if the relationship between the number of the panels and the cost of panels is proportional. Explain your reasoning.

Mathematical Note

Linear relationships in the form y = mx + b are not proportional.

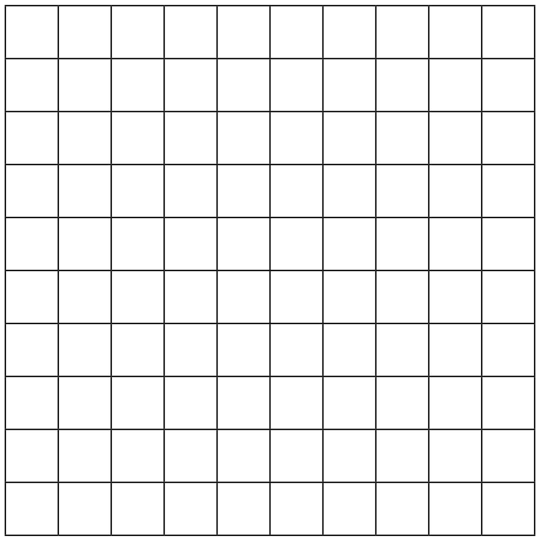
For example, let's consider the equation y = 2x.
If x = 1, then y = 2 and if x = 4, then y = 8.
The proportional relationship ½ = 4/8 is true.
Therefore the equation y = 2x is a linear proportional relationship.

Now let's consider the equation y = 2x + 3.
If x = 1, then y = 5 and if x = 4, then y = 11.
Therefore the proportional relationship 1/5 = 4/11 is false.
Therefore the equation y = 2x + 3 is a linear non-proportional relationship.

3. Let *p* represent the number of solar panels and *c* the cost of panels. The table below shows the number of array panels and the cost of panels. With the ratio from above, use a proportion to complete the table below with the given information.

Photovoltaic Modules								
Number of panels (*p*)	0	2	4	6	8	10	60	*P*
Cost of panels in millions (*c*)								

4. Graph the relationship between the number of panels and the cost of panels on the coordinate grid provided.



5. Write an equation that describes the relationship between the number of panels and the cost of panels.

6. What will the cost be for the eight panels needed for the ISS? Use a proportion and the equation to find this cost.

Using a Proportion	Using an Equation

7. How many panels could the ISS buy with $115 million?

Use the multiple representations to find your solution.

Using a Proportion	Using the Equation	Using the Table	Using the Graph

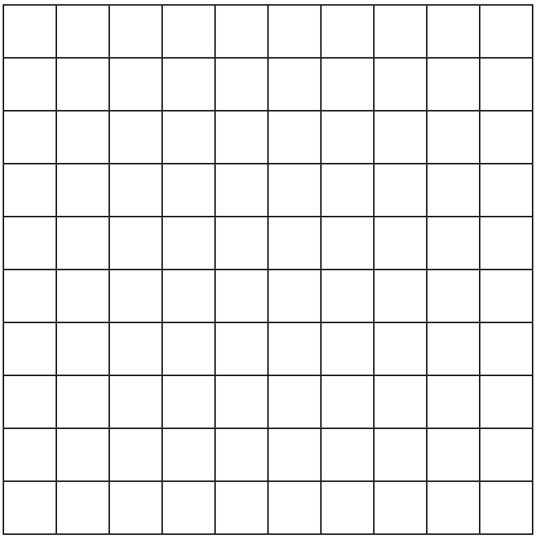
8. Compare the solutions under each of the multiple representations.

9. Let p represent the number of solar panels and t the total cost of panels. The Mercury Solar Company adds a $3 million fee to any purchase. This fee pays the research and development expenses attached to the manufacturing of the panels.

The table below shows the number of array panels and the total cost of panels. Complete the table.

Photovoltaic Modules								
Number of panels (*p*)	0	2	4	6	8	10	60	*P*
Total cost of panels (*t*)								

10. Graph the relationship between the number of panels and the total cost of panels on the coordinate grid provided.



11. Write an equation that describes the relationship between the number of panels and the total cost of panels.

12. What will the total cost be for eight panels needed for the ISS? Use a proportion and the equation to find this cost.

Using a Proportion	Using an Equation

13. How many panels could the ISS buy with $115 million?

    Use the multiple representations to find your solution.

Using a Proportion	Using the Equation	Using the Table	Using the Graph

14. Write the equation for the relationship between the number of array panels and the cost of panels and the equation for the relationship between the number of array panels and the total cost of panels.

______________________ Panels vs. Cost of Panels

______________________ Panels vs. Total Cost of Panels

15. Compare the multiple representations of #7 and #13. How are the equation, table, and graph of these two relations similar and different?

16. List the characteristics you have encountered of a proportional and a non-proportional relationship for each of the representations.

Multiple Representation	Proportional Relationship	Non-Proportional Relationship
Proportion		
Equation		
Table		
Graph		

17. Determine the proportionality of the relationship between the number of array panels and the cost of panels. Refer to the equation, table, and graph of the relation to support your answer. Use as many characteristics as possible from your list above.

18. Determine the proportionality of the relationship between the number of array panels and the total cost of panels. Refer to the equation, table, and graph of the relation to support your answer. Use as many characteristics as possible from your list above.

You are now going to make a model of these solar panels. Your setup materials will cost you fifteen dollars. The paper to make the model panels will cost you four dollars per a set of panels. You plan to sell these models to the NASA for seven dollars per set.

19. What relationship would be linear and proportional?

20. Justify your response.

21. What relationship would be linear and non-proportional?

22. Justify your response.

23. Describe how the equation of the relationship in problem 19 could help you determine its proportionality.

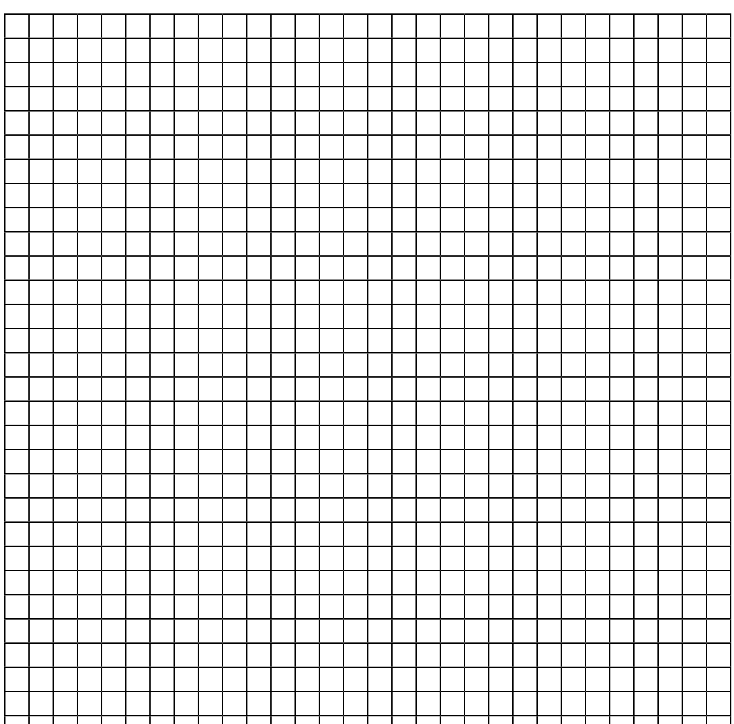
ATLANTIS MISSION THREE

Non-Proportional Relationships

1. Recall that the cost of four panels is $75 million. The Mercury Solar Company adds a $3 million fee to all purchases. This fee pays the research and development expenses attached to the manufacturing of the panels. The table below shows the number of array panels and the total cost of panels. Let *p* represent the number of solar panels and *c* the total cost of panels. Complete the table.

Number of panels (*p*)	Process Row	New cost of panels in millions (*c*)	Rate of cost to panels (*c/p*)
0			
2			
4			
6			
8			
10			
20			
. . .			
P			

2. Graph the relationship between the number of panels and the new total cost of the panels on the coordinate grid provided.



3. Explain how you found the total cost of the panels.

4. Write an equation that describes the relationship between the number of panels and the total cost of panels.

5. Verify the total cost for the eight panels needed for the ISS using the equation. Use the equation to find the cost of fifteen panels.

Using your Equation for 8 panels

Using your Equation for 15 panels

6. The ISS is considering storing panels for future use. What will be the cost for purchasing 77 panels?

7. Can you use a proportion to find the total cost for the eight panels? Explain why or why not.

Using a Proportion

8. Compare the solution using an equation and the solution using a proportion.

Mathematical Note

In the equation c=mp + b, the relationship between c and p is not proportional. On the other hand, the rate of change between the cost per panel (Δc) and the number of panels (Δp) is proportional.

9. Show the multiple representations for the relationship between the number of panels and the new total cost of panels in this mission.

Using your Equation	Using the Table	Using the Graph	Proportion

10. Is the constant rate of change in your equation also a constant of proportionality? Explain your opinion.

11. In Mission One, the cost of four panels was $75 million. Show the multiple representations from Mission One in the space below.

Equation	Table	Graph	Proportion

12. Is the constant rate of change in the equation in Mission Two also a constant of proportionality? Explain your opinion.

13. Write the equation for the relationship between the number of panels and the cost of panels, and the equation for the relationship between the number of panels and the new total cost of panels.

    The Number of Panels to the Cost of Panels ______________________

    The Number of Panels to the Cost of Panels including an additional fee ______________________

14. If the cost per panel is the same for both situations, why is the total cost different?

15. Compare multiple representations for Mission Two and Mission Three. How are the equations, tables, and graphs of these two relationships similar and how are they different?

Representation	Similarities	Differences
Equation		
Table		
Graph		

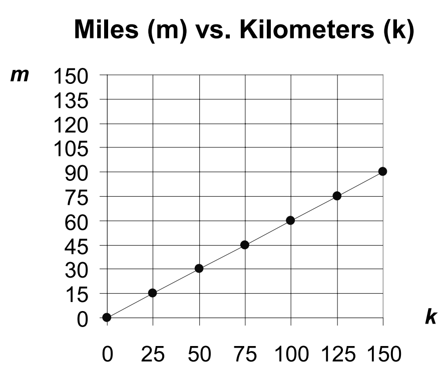
16. Which relationship is proportional? Justify your response.

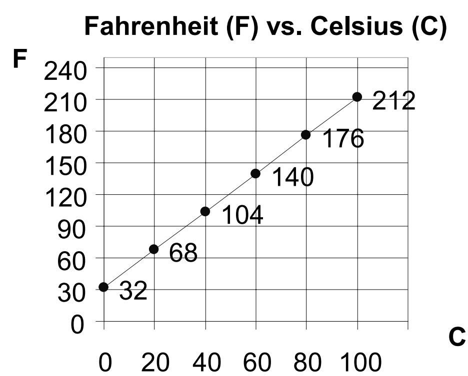
17. Which relationship is not proportional? Justify your response.

ATLANTIS MISSION FOUR

Proportional & Non-proportional Relationships

The International Space Station, ISS, uses different systems to record data for each mission. It is important to be able to convert these measurements between systems as different countries use different systems. The International Space Station records their measurements in a variety of systems. Let us take a look at the two graphs below that show the relationship of length measurements and temperature between the metric and customary system. Recall the following conversion: one mile is approximately 1.6 kilometers.





4. Complete the following tables by reading the graphs above. Let *k* represent the number of kilometers and *m* the number of miles.

<table>
<tr><td>Change in kilometers</td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td><td></td></tr>
<tr><td></td><td colspan="2">^</td><td colspan="2">^</td><td colspan="2">^</td><td colspan="2">^</td><td></td></tr>
<tr><td>Number of kilometers (k)</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>. . .</td><td>k</td></tr>
<tr><td>Number of miles (m)</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="2">v</td><td colspan="2">v</td><td colspan="2">v</td><td colspan="2">v</td><td></td></tr>
<tr><td>Change in miles</td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td><td></td></tr>
<tr><td>$\frac{\text{change in } m}{\text{change in } k}$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Ratio $\frac{m}{k}$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Let C represent the degrees in Celsius and F the degrees in Fahrenheit.

Change in kilometers									
	^		^		^		^		
Number of degrees Celsius, C								. . .	C
Number of degrees Fahrenheit, F									
	v		v		v		v		
Change in degrees Fahrenheit									
$\frac{\text{change in } F}{\text{change in } C}$									
Ratio $\frac{F}{C}$									

5. Describe your rule for finding the number of miles when the number of kilometers is given.
6. Explain how to convert 200 kilometers?
7. Show how you could use the unit rate of miles to kilometers to find the conversion of 200 kilometers into miles.

   Unit rate = ____________

8. Use your rule to write an equation that describes the relationship between the number of kilometers, k, and the number of miles, m.
9. If a degree of 50 degrees Celsius was recorded, explain how you could convert it into degrees Fahrenheit?
10. Show how you could use a rate of degrees Celsius to Fahrenheit to convert 100 degrees Celsius into degrees Fahrenheit.

    Unit rate = ____________

11. Use your rule to write an equation that describes the relationship between the degrees in Celsius, *c*, and the degrees in Fahrenheit, *f*. Using the two graphs, tables, and equations of two conversion relationships complete the following information:

Differences in graphs	
Differences in table	
Differences in equations	

12. Given what you have learned about proportional and non-proportional relationships, describe the two conversion relationships in terms of their equation, table, graphs, and constant rates.

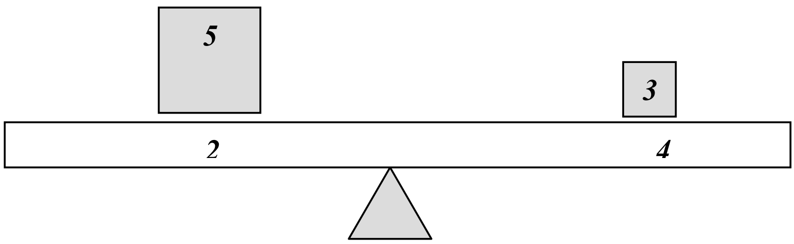
<u>Kilometers to Miles</u>

<u>Celsius to Fahrenheit</u>

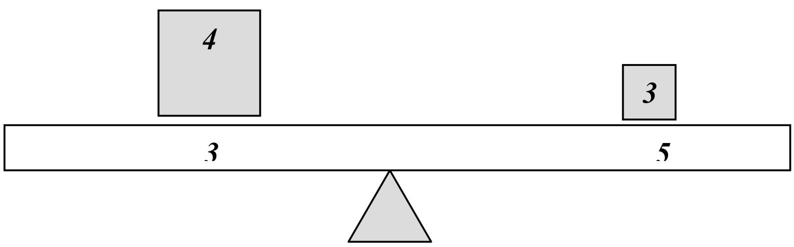
GENERALIZATION AND EXTENSION

1. Each scale below has numerical values for weight and distance on both sides of the scale. The triangle in the middle is a fulcrum. Which scale below is balanced?

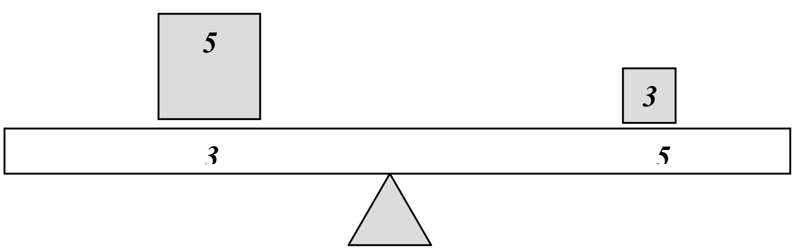
A.



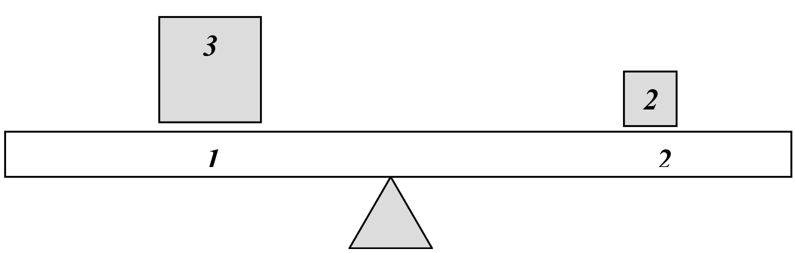
B.



C.

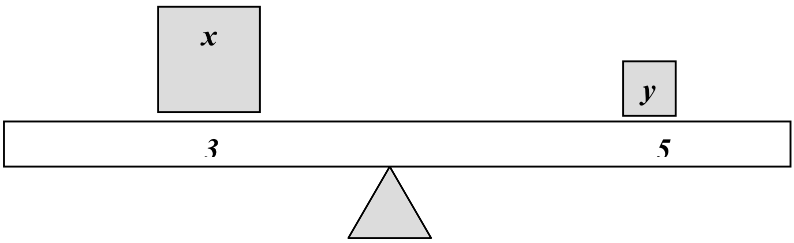


D.



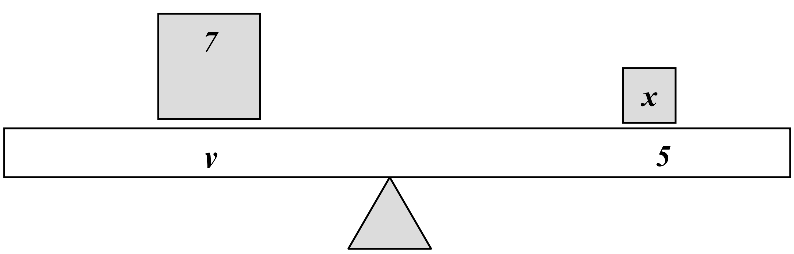
Explain your choice below:

2. The scale below has a weight of *x* units at a distance of *3* units — on the left side of the scale, and a weight of *y* units at a distance of *5* units — on the right side of the scale.



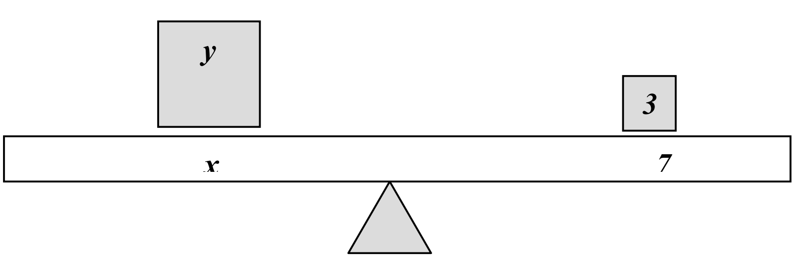
If the scale is balanced, what is the relationship between weights *x* and *y*? Write down the relationship as an algebraic expression. Describe a type of this relationship. Provide your solution in the space below:

3. The scale below has a weight of *x* units at a distance of *5* units — on right side of the scale, and a weight of *7* units at a distance of *x* units — on left side of the scale.



If the scale is balanced, what is the relationship between weight *x* and distance *y*? Write down the relationship as an algebraic expression. Describe the type of this relationship. Provide your solution in the space below:

4. The scale below has a weight of *y* units at a distance of *x* units — on the left side of the scale, and a weight of *3* units at a distance of *7* units — on the right side of the scale.

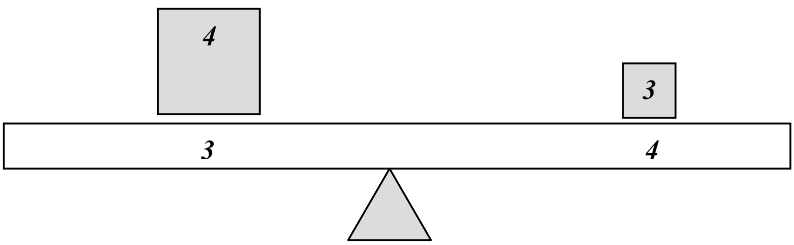


If the scale is balanced, what is the relationship between weight y and distance x? Write down the relationship as an algebraic expression. Describe a type of this relationship. Provide your solution in the space below:

Mathematical Note

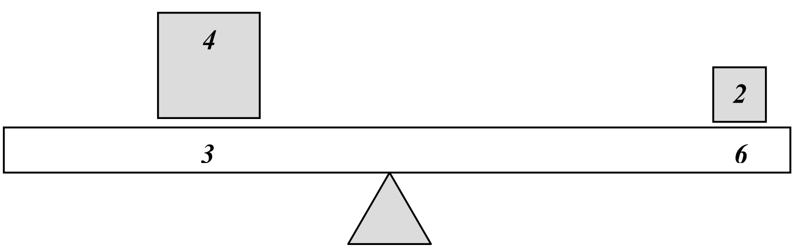
While performing exercises 1-4 you, probably, noticed that multiplicative structures could be balanced and unbalanced.

The multiplicative structure below is balanced because 4x3=3x4.



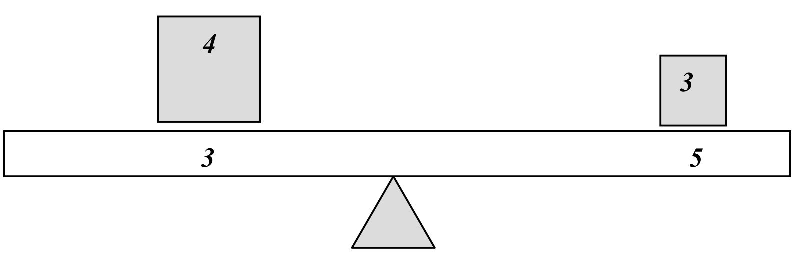
In general, this type of the ***multiplicative balance*** $ab=ba$ we call ***commutative.***

The multiplicative structure below is balanced too because 4x3=2x6.

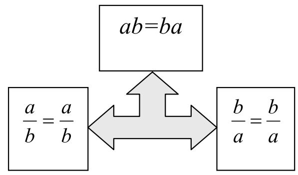


In general, this type of the multiplicative balance $ab=cd$ we call ***non-commutative***.

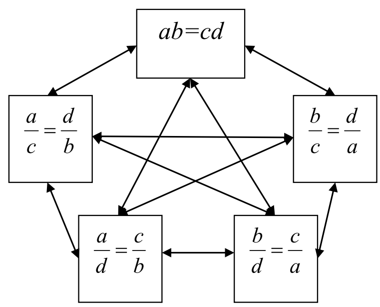
The multiplicative structure below is unbalanced because 4x3≠3x5.



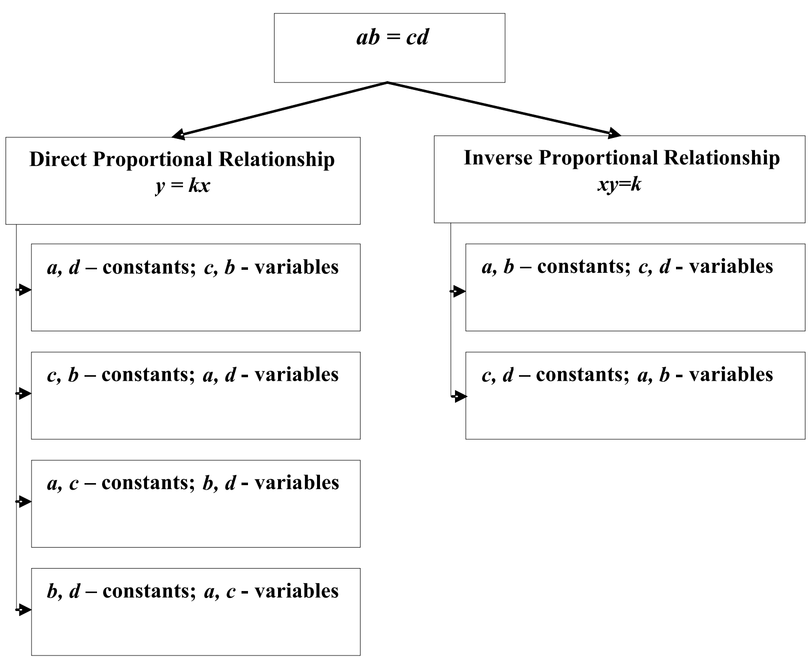
The commutative multiplicative balance $ab=ba$ can be represented in the following equivalent forms:



The non-commutative multiplicative balance $ab=cd$ can be represented in the following equivalent forms:



The non-commutative multiplicative balance $ab=cd$ can lead to two types of proportional relationship - ***direct and inverse -*** depends on which terms of the balance are fixed (***constants***) and which terms vary (***variables***).



Let's consider the first setting for the direct proportional relationship when a and d are constants and b and c are variables. Let the variable b be equal to y and let the variable c be equal to x. So, $b=y$, $c=x$. Then the multiplicative balance statement $ab=cd$ can be rewritten as $ay=xd$. Let's solve this statement for y:

$$y = \frac{d}{a}x.$$

Since both a and d are constants, then the ratio of two constants $\frac{d}{a}$ is a constant too. Let constant k equal to the *ratio* $\frac{d}{a}$ then $k=\frac{d}{a}$. Finally, we have: $y=kx$.

5. Show how the second setting for the direct proportional relationship when b and c are constant and a and d are variables leads to the algebraic statement $y=kx$.

6. Show how the third setting for the direct proportional relationship when a and c are constant and b and d are variables leads to the algebraic statement $y=kx$.

7. Show how the fourth setting for the direct proportional relationship when b and d are constant and a and c are variables leads to the algebraic statement $y=kx$.

Let's consider the first setting for the inverse proportional relationship when a and b are constants and c and d are variables. Let us denote symbol x to the variable c and symbol y to the variable d: $c=x$, $d=y$. Then the multiplicative balance statement $ab=cd$ can be rewritten as $ab=xy$. Since both a and b are constants, the product of two constants ab is a constant too. Let's denote symbol k to the *product* ab: $k=ab$. Finally, we have: $k=xy$ or $xy=k$.

8. Show how the second setting for the inverse proportional relationship when c and d are constant and a and d are variables leads to the algebraic statement $xy=k$.

Mathematical Note

In both cases of direct and inverse proportionality k is called ***a constant of proportionality.*** The difference in the role of k between direct proportional and inverse proportional relationship is the following:

- in the inverse proportional relationship $\boldsymbol{xy=k}$, k represents *a coefficient;*
- in the direct proportional relationship $\boldsymbol{y=kx}$, k represents *a coefficient* with a special role — ***a constant rate of change.***

DEBUGGING MISCONCEPTIONS

In the table below complete the column "How to fix the misconception?"

Type of misconception	Example of Misconception	Cause of Misconception	How to Fix the Misconception?
1. *Ignorance of data*	On a missing value problem, a student uses only two of the three pieces of the given information to find a solution. Using the gum example, a student might say 8 sticks of gum costs 72 cents since 9 cents times 8 sticks is 72, not using the piece of data which tells us that 9 cents is for 2 sticks of gum.	*Students ignore part of the question or some of the data.*	
2. *Additive reasoning in proportional situation*	This strategy is called a *constant difference*. Here, a student might notice that 9 cents is 7 more than 2 sticks, so to find the missing value they would add 7 to 8, the number of sticks of gum in the second ratio, to predict that 8 sticks would cost 15 cents.	*Students use an additive strategy that focuses on the constant different between two numbers in a proportion, as opposed to their constant rate.*	
3. *Mixing additive and multiplicative strategies*	Here, a student may find the correct non-integer ratio, but only multiplies by the whole number component of the ratio to find the missing value, adding the remainder. For the gum problem, a student might say that 9/2 is 4 R1, so they would multiply 8 by 4 and then add 1, getting 33 instead of 36.	*Students may use a combination of additive and multiplicative strategy, often occurring with non-integer ratios.*	
4. *False rate or proportion*	Students may set up a proportion, but put the numbers in the wrong places. They may find the wrong unit rate, but use it correctly to find the missing value.	*Students may use a faulty application of a correct strategy.*	
5. *False assumption*	For example, in the Mr. Short and Mr. Tall problem, the student may assume that the little paper clips are half the size of the big paper clips. Therefore, they choose a scale factor of two and make a prediction accordingly. In actuality, the scale was only 1.5 for the task (remember, 4/6 = 6/x).	*Students make a change in scale that they predict the given information not given in the problem.*	
6. *False numerical preference*	Students may say that they feel one ratio is larger than the other because they like number 5, and not because of the values of two ratios.	*Students make a decision based on the appearances of other extraneous factors in a problem, often called 'using a qualitative method'.*	

APPLICATION

Use the following information to answer questions 1 — 9 below.

Andy and Angela are remodeling their kitchen. The first phase will involve replacing all countertops. They have chosen a countertop that will cost them $350 for every 10 square feet including installation. To make sure all permits are filed and the work is up to code, the contractor charges an additional $1,500 to oversee the project.

Andy and Angela decide to install as much of the countertop around the kitchen as their budget will allow. The contractor has given them four designs with different amounts of square feet for each of the four designs. Fill in the rows of table for designs 1-4 to help you answer the questions that follow:

Cost					
	s = square feet	Countertop cost	Contractor's project oversee charge	*t* = total cost	*(s, t)*
Design 1	200				
Design 2	175				
Design 3	150				
Design 4	125				
Andy and Angela's design					

1. How much will the countertop cost for design 1?

   What is the total cost if they decide to go with design 1?

2. How much will the countertop cost for design 2?

   What is the total cost if they decide to go with design 2?

3. How much will the countertop cost for design 3?

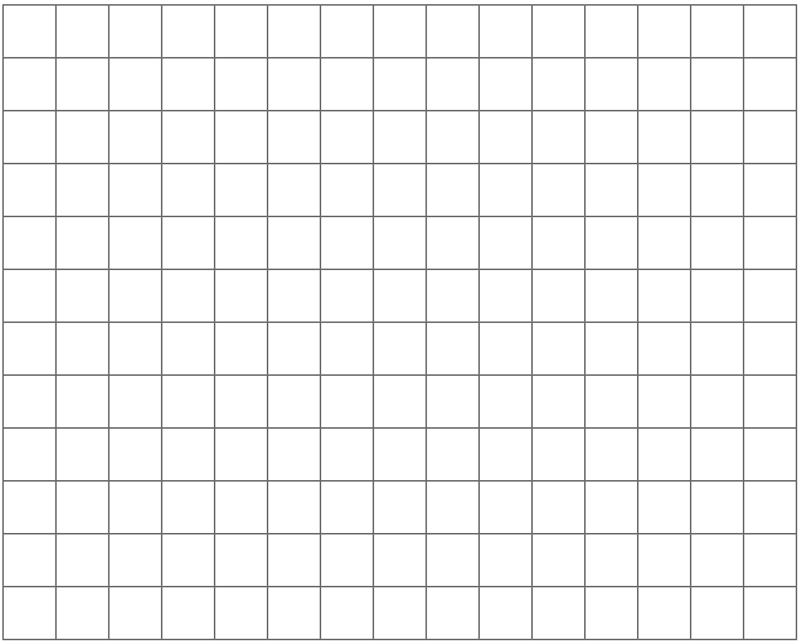
   What is the total cost if they decide to go with design 3?

4. How much will the countertop cost for design 4?

   What is the total cost if they decide to go with design 4?

5. Let s = square feet and t = total cost

Use the grid below to plot the points (*s*, *t*) collected from each design.



6. If Andy and Angela have $6,500 to spend on buying and installing new countertops, estimate how many square feet of countertop they could have installed if they come up with a new design? (It may not be one of the options given.)

7. Fill in the last row of the table using the square feet you have determined from the graph. Check your answer.

8. Explain in words, how you would determine the total cost for any number of square feet?

9. Write an equation using s for square feet and t for total cost that expresses the explanation you just wrote.

10. It is recommended that an average adult take a daily allowance of 0.8 grams of protein for every 2.2 pounds of body weight. Harry weighs 160 pounds.

    a. How many grams of protein should Harry allow himself?

    b. Write an equation anyone could use to determine their daily grams of protein allowance using w — as the number of pounds they weigh and p — as the daily amount of protein they need in grams.

    c. Can this equation be expressed as a proportion? Is the relationship a proportion?

11. Harry's sister, Angela is on a swim team and needs an additional 5 grams of protein each day on top of the recommended average daily allowance. She weighs 128 pounds.

    a. How many daily grams of protein should Angela allow herself?

    b. Write an equation Angela's swim team members can use to determine their daily grams of protein allowance using p as the number of pounds they weigh.

    c. Can you write the equation for a swimmer as a proportion? Is the relationship between pounds and grams of protein needed for a swimmer a proportional relationship?

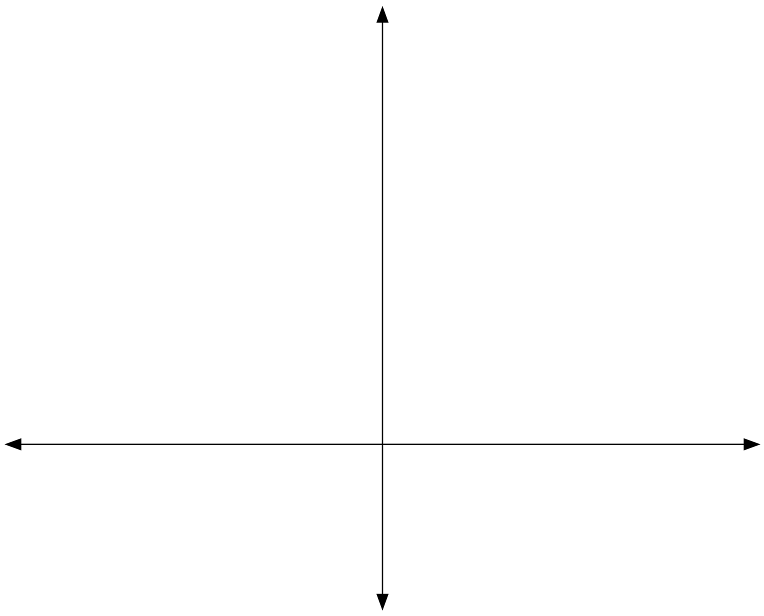
12. Complete the following data for Angela's swim team members. Use your work from 11 above to complete the table for the other swim team members.

Team member	Weight in pounds, *w*	Grams of protein needed based on weight	Additional grams needed for swimmers	Total daily allowance of protein needed, *p*	*(w, p)*
Angela	128				
Isabella	145				
Daisy	130				
Laura	152				
Ana	125				
Dolly	141				
Any team member	*w*				

13. Explain the process you used to fill in the column for total daily allowance of protein needed.

14. Write the process you used in the form of an equation. Use w for swimmer's weight and p for the amount of protein needed. Use this to fill in the last row of the table.

15. In the coordinate x, y — plane given below, plot the points from the table you completed for Angela's team members. Use weight for the horizontal axis and protein for the vertical axis.



16. Connect the points to form a line.

17. Plot the point for Harry's weight and protein allowance. Where would the line for all the non-swim team members like Harry go?

Guided Review of Proportion Concepts

1. The heart of an average adult beats 16 times every 12 seconds.

   a. Is this a proportional relationship?

   b. What is the unit rate of the heartbeat of an average adult?

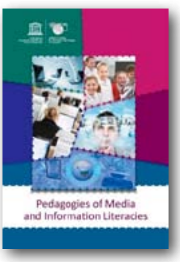
   c. How many times does the average adult heart beat in one minute?

2. Forming a Concept and Organizing Knowledge.

   $\frac{a}{b} = \frac{c}{d}$ is a true proportion. Write the proportion in at least two other ways.

RECENT IITE PUBLICATIONS

Information and Communication Technologies in Education (In Russian)
UNESCO IITE, Moscow, 2013.
The monograph summarizes the results obtained within UNESCO IITE projects and analytical materials provided by IITE experts. The book covers the best practices of UNESCO Member States in integration of ICT in their educational systems in the context of development of knowledge society and emergence of new technologies and alternative forms of education delivery. Based on the generalisation and extrapolation of contemporary trends in ICT in education the book presents recommendations for education policy.

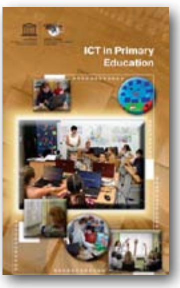


Pedagogies of Media and Information Literacies
UNESCO IITE, Moscow, 2012.
IITE produced this Handbook in collaboration with the Finnish Society on Media Education. The Handbook published in English and Russian should help teachers to enhance their media and information literacy and encourage them to take up media education in the classroom. The main target group is teachers of secondary schools who are either in training or in service.

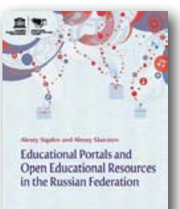
IITE-2012 International Conference "ICT in Education: Pedagogy, Educational Resources and Quality Assurance"(Abstracts)
UNESCO IITE, Moscow, 2012.
The Book of Abstracts of the IITE-2012 International Conference (organized by UNESCO IITE and UNESCO Moscow Office in cooperation with the Moscow State University of Economics, Statistics and Informatics and State Institute of Information Technologies and Telecommunications "Informika" on 13-14 November 2012 in Moscow, Russia) includes keynote speeches and extended abstracts in English or Russian.

Recognizing the potential of ICT in early childhood education (in Russian)
UNESCO IITE, Moscow, 2012.
The analytical survey published by IITE in 2010 in English has been translated into Russian and published. The survey suggests strategies for the development of ICT capability of ECE centres and recommendations, which should be helpful for educators, parents and school policy decision makers in their efforts to adapt the child development process to the continuous evolution of the digital universe.

Media and Information Literacy: Curriculum for Teachers (in Russian)
UNESCO IITE, Moscow, 2012.
The Institute has translated into Russian and published the UNESCO curriculum for teachers "Media and Information Literacy". Authors: Wilson, Carolyn; Grizzle, Alton; Tuazon, Ramon; Akyempong, Kwame; Cheung, Chi-Kim. Scientific editors of the Russian version: Prof. Natalia Gendina, Director of the Research Institute of Information Technologies, Kemerovo State University; Prof. Sergey Korkonosenko, Chair of the Department of the Theory of Journalism and Mass Communications, St.Petersburg State University.



ICT in Primary Education
UNESCO IITE, Moscow, 2012.
The book is the first publication in the framework of the UNESCO IITE three-year project. It explores the origins, settings and initiatives of effective integration of up-to-date innovative technologies in primary school and accumulates the best practices of ICT incorporation gathered from the project sample primary schools across the globe. The analytical study was elaborated by IITE experts from Chile, Hong Kong, Hungary, Slovak Republic, South Africa, Russia, UAE and UK.

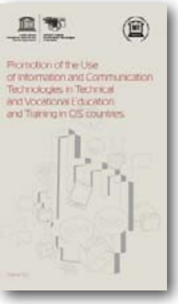


Educational Portals and Open Educational Resources in the Russian Federation
UNESCO IITE, Moscow, 2012.
The survey "Educational Portals and Open Educational Resources in the Russian Federation" (authors: Alexey Sigalov and Alexey Skuratov, INFORMIKA) provides an up-to-date survey of the current level of development of educational materials and repositories of educational resources in Russia.

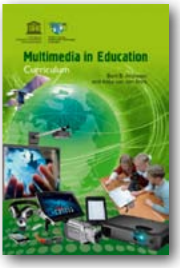
Open Educational Resources in Poland: Achievements, Challlenges and Prospects for Development
UNESCO IITE, Moscow, 2013.
IITE published a new case study describing main achievements, challenges and prospects for development of electronic textbooks and Open Educational Resources in Poland developed by Kamil Śliwowski and Karolina Grodecka (Coalition for Open Education and Akademia Górniczo-Hutnicza im. Stanisława Staszica).

Engineering of Learning:
Conceptualizing e-Didactics
UNESCO IITE, Moscow, 2013.
The book opens the series of UNESCO IITE research in digital pedagogy. The author (Mourat Tschoshanov) summarizes new tendencies in the development of pedagogical research under the rapid introduction of ICTs in education. The main focus of the book is design, development, implementation, and assessment of learning experiences through the use of ICT in various formats: face-to-face, blended, and distance education.

Information and Communication Technologies for Visually Impaired People
UNESCO IITE, Moscow, 2012.
The training course was developed in cooperation with the Institute of Professional Rehabilitation "REHACOMP". The course covers pedagogical, technical and practical aspects of education and training management of visually impaired people through advanced integration of standard and specialized innovative technologies.



Analytical Report "Promotion of the Use of ICTs in TVET in CIS Countries"
UNESCO IITE, Moscow, 2012.
The report is prepared by IITE within the framework of the joint project with the Intergovernmental Foundation for Educational, Scientific and Cultural Cooperation of CIS. The publication presents the results of comparative studies on the current situation and major tendencies in ICT use in TVET in the Kyrgyz Republic, Republic of Armenia, Republic of Azerbaijan, Republic of Belarus, Republic of Kazakhstan, Republic of Moldova, Republic of Tajikistan, Republic of Uzbekistan, Russian Federation and Ukraine.



Multimedia in Education Curriculum
UNESCO IITE, Moscow, 2013.
This is a revised edition of the book prepared by Bent B. Andresen (Danish University of Education, Denmark) and Katja van den Brink (University of Landau, Germany) previously published by IITE in 2001. The curriculum provides a well-structured and systematic explanation of several pedagogical scenarios for the use of multimedia in education, including the description of the different aspects of performance and portfolio assessment, the role of multimedia end users, multimedia production process, practical use of multimedia in teaching and learning.

UNESCO IITE International Master Programme "ICT in Teacher Professional Development" (Russian Version)
UNESCO IITE, Moscow, 2013.
This Master Programme Curriculum aims at bringing together key ICT awareness, skills, knowledge and attitudes, and build upon them an efficient strategy. The Russian version of the Master Programme is developed according to the requirements of the third generation of the Russian Federation Standards and addressed to Russian and CIS universities.

Guidelines on UNESCO ICT-CFT Adaptation (Methodological Approach to Localization of the UNESCO ICT-CFT)
UNESCO IITE, Moscow, 2013.
The Guidelines contain a description of the methodological approach to localization of the UNESCO ICT-CFT and aim to assist UNESCO Member States developing national (regional) ICT competency standards for teachers, standard setting being one of the key components of the policy in the field of ICT application and professional development in education.

IITE Policy Briefs

Technology-Enhanced Assessment in Education
UNESCO IITE, Moscow, 2012.
ICTs offer new opportunities for innovation in educational assessment. Technology-enhanced assessment includes strategies for self-assessment and peer assessment emphasizing the next steps needed for further learning. The Policy Brief contains an overview of the state-of-the-art, major trends, challenges and policy recommendations on design, implementation and monitoring of ICT-based assessment.

ICTs in Global Learning / Teaching / Training
UNESCO IITE, Moscow, 2012.
The Policy Brief outlines the systematic approach that must be taken for ICTs to fulfill their promise. Schools must focus on training teachers; creating curriculum materials; making organizational arrangements; and creating networks. Higher education must rethink its role for a world where open educational resources make high quality content abundant and students want to combine work and study seamlessly.

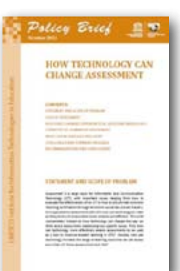
Personalized Learning: A New ICT-Enabled Education Approach
UNESCO IITE, Moscow, 2012.
Personalized learning becomes increasingly prominent in policy discussions on the future of education. The latest developments in ICT technologies and digital content are revolutionizing education. They make the benefits of personalized learning available for mass adoption in schools, universities and adult training institutions. The Policy Brief highlights the advantages of personalized learning and offers the ways of implementing it in educational institutions.

ICTs for Curriculum Change
UNESCO IITE, Moscow, 2012.
The emergence of new jobs and the change in existing jobs due to ICTs has an impact on the curriculum of vocational education programs. The potential of ICTs as a medium for teaching and learning is recognized by many, but the implementation is often problematic, as a result relatively few students worldwide are offered the opportunity to learn with the help of ICTs. To effectively integrate ICTs into educational practices teachers need to develop competencies which will help them to integrate appropriate pedagogy and knowledge about ICTs.

ICT and General Administration in Educational Institutions
UNESCO IITE, Moscow, 2012.
This policy brief provides an overview of ICTs that are used to support educational administration. This includes technologies used to support learners from their initial inquiries about courses through to graduation, technologies to support teachers in the design and delivery of teaching and technologies to support the research lifecycle from bidding through to project management and finally research dissemination.

Quality management and assurance in ICT-integrated pedagogy
UNESCO IITE, Moscow, 2012.
'Quality management and assurance' (QA) is the process of ensuring that systems, establishments, practices and resources are 'fit for purpose'. The Policy Brief draws attention to some of the new ICT-linked issues involved that require additional QA criteria. One of the most important issues is the impact on the professional development of teachers and lecturers. Another is the potential of ICT to enhance education, which requires flexibility.

Alternative Models of Education Delivery
UNESCO IITE, Moscow, 2012.
The goal of this Policy Brief is to produce a number of alternative models of education delivery in the formal education sector. Five alternative models would sufficiently populate the various subsectors of formal education. The models have to be "archetypal", easy and quick to describe, memorable, repeatable, and translatable into other languages. They should also be generalisable, scalable, sustainable, deployable, and deliverable in most high-income economies. For each model the features, the advantages and the disadvantages are outlined.



How Technology Can Change Assessment
UNESCO IITE, Moscow, 2012.
Many discussions of technology-based assessments concentrate on automating current methods of testing to save time and cost. However, technology also changes what educators can assess, how, when and for what purpose. Assessments can be embedded in ICTs, and have the potential to measure learning processes, in addition to end-of-lesson knowledge. Technology-aided assessments are useful not only in the evaluation of ICTs, but also as part of the design process, leading to iterative improvement.

Learning Analytics
UNESCO IITE, Moscow, 2012.
Learning Analytics is a rapidly growing research field, with presumably disruptive potential. While educational researchers have for many years used computational techniques to analyze learner data, generate visualizations of learning dynamics, and build predictive models to test theories — for the first time, these techniques are becoming available to educators, learners and policy makers.

ICTs in Early Childhood Care and Education
UNESCO IITE, Moscow, 2012.
The Policy Brief outlines the values that ICT offers to early childhood learning; gives different perspectives on the process of implementation of ICTs into ECCE practice, lists most frequent safety concerns and presents general criteria for determining the developmental appropriateness of the ICT tools to be applied in ECCE. The message is the understanding that the potential of ICT for ECCE can be productively harnessed only if new technologies are integrated into early childhood learning experience alongside many other ordinary everyday activities.

Introduction to MOOCs: Avalanche, Illusion or Augmentation?
UNESCO IITE, Moscow, 2013.
The New York Times labeled 2012 'The Year of the MOOC'. Less than 24 months after the launch of the first massive open online course (MOOC) at Stanford University and with potentially over 5 million students around the world now registered with a MOOC platform, massive open online courses appear to be a new and significant force within higher education. This Policy Brief provides a background to the expansion of MOOCs, explain their differences and similarities, identify the types of students using MOOCs, investigate their business models and potential direction, and finally scope the risks and benefits associated with their development.

Mourat Tchoshanov was born and educated in Turkmenistan. He received his Ph.D. in Russia. Currently Prof. Tchoshanov lives and works in the U.S. He is dual appointed as a Professor of Mathematics Education by the Department of Mathematical Sciences and the Department of Teacher Education at the University of Texas at El Paso (UT-El Paso). He is also in charge of the Ph.D. Program in Teaching, Learning, and Culture at UT-El Paso. Areas of his research interest include but are not limited to e-Didactics, didactical engineering, teacher knowledge, representation and cognition in learning. The list of his publications includes more than 150 scholarly works, 10 of which are monographs printed by national and international publishers.

ISBN 978-5-905385-14-8
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