Educational Practices Series

30

Proportional reasoning

by Wim Van Dooren, Xenia Vamvakoussi, and Lieven Verschaffel

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The Series was started in 2002, as a joint venture between the International Academy of Education (IAE) and the International Bureau of Education (IBE). So far 30 booklets have been published in English and many of them have been translated in several other languages.

The success of the Series shows that the booklets meet a need for practically relevant research-based information in education.
The series is also a result of the IBE's efforts to establish a global partnership that recognizes the role of knowledge brokerage as a key mechanism for improving the substantive access of policymakers and diverse practitioners to cutting-edge knowledge. Increased access to relevant knowledge can also inform education practitioners, policymakers and governments how this knowledge can help address urgent international concerns, including but not limited to curriculum, teaching, learning, assessment, migration, conflict, employment and equitable development.

Governments need to ensure that their education systems meet their core and indisputable mandate, which is to promote learning and, ultimately, to produce effective lifelong learners. With the aggressive pace of contextual change in 21st century, lifelong learning is a critical source of adaptability, agility to adapt, and the resilience required to meet challenges and opportunities. Yet, for many countries around the world, effective facilitation of learning remains a daunting challenge. Learning outcomes remain poor and inequitable. Intolerably high proportions of learners fail to acquire prerequisite competences for lifelong learning such as sustainable literacy, digital literacy, critical thinking, communication, problem solving, as well as competences for employability and for life. Systems' failure to facilitate learning co-exists with impressive advancements in education research, driven by research from diverse fields, including the sciences of learning, particularly the neuroscience of learning, and advancements in technology.

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The IBE recognizes the advancements already made, but also that there is still much more work to be done. This can only be achieved through solid partnerships and a collaborative commitment to building on previous lessons learned and continued knowledge sharing.

The Educational Practices booklets are illustrative of these ongoing efforts, by both the International Academy of Education and the International Bureau of Education, to inform education policymakers and practitioners on the latest research, so they can better make decisions and interventions related to curriculum development, teaching, learning and assessment.

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Introduction

The middle school math curriculum (grades 5 to 8) contains many important mathematical notions. Still, proportionality can be considered among the most important ones. Mathematics educators suggest that the ability to reason proportionally deserves whatever time and effort that educators and students must invest to assure its development.

Proportionality is the capstone of elementary arithmetic, number, and measurement concepts, and at the same time one of the most elementary understandings one needs for more advanced mathematics. Understanding proportionality is not only essential for comprehending higher level mathematics such as geometrical similarity or probability, it is also most useful for everyday life.

The development of proportional reasoning is a complex process that progresses gradually over many years. However, despite the attention teachers pay to this area in the curriculum, students continue to experience many difficulties with it. In this research guide, we will address some of the most important difficulties and the ways to deal with them.

Proportional reasoning is situated in the multiplicative field. Two measure spaces are involved that are modelled by a linear function; i.e., a function of the form $f(x) = ax$. Consider the following example: When she makes strawberry jam, my grandmother uses 3.5 kg of sugar for 5 kg of strawberries. How much sugar does she need for 8 kg of strawberries? This example can be schematically represented as follows:

M1	M2
a	b
c	x

with the values a (5) and c (8) belonging to a first measure space M1 (strawberry weights), and b (3.5) and x (unknown) belonging to a second measure space M2 (sugar weights).

Proportional reasoning refers to the ability to understand, construct, and use the multiplicative relationship between the two co-varying measure spaces (which is called “functional reasoning”; see below) or within the measure spaces (called “scalar reasoning”; see below). This typically implies the multiplication and division operations, but, as will become clear throughout this guide, students can also fruitfully apply addition and subtraction to express and handle multiplicative relations.

The closely related term “multiplicative reasoning” is often used to refer to the less advanced types of reasoning required to solve simple, one-step multiplication and division problems, such as: Richard buys 4 cookies priced at 15 cents each. How much does he have to pay? Such problems are simple cases of proportional situations, as one of the four terms involved is equal to one, allowing one to solve the problem by a single multiplication.

Researchers generally distinguish two major types of proportional problems: missing value, and ratio comparison. We can consider the strawberry jam problem (given above) a missing-value problem in the sense that three of the four values in the proportion are given and the fourth one has to be calculated. One can turn it into a ratio comparison problem by changing it to: Yesterday, my grandmother made strawberry jam using 3.5 kg of sugar for 5 kg of strawberries. Today, she used 6.5 kg of sugar for 8 kg of strawberries. Which jam tasted sweeter?

In this research-based practice guide, we focus on missing-value problems, as these have received most attention in research. However, one could easily transfer most of the findings and educational implications to ratio comparison problems.

Suggested readings: Lesh, Post, & Behr, 1988; NCTM, 1989; Vergnaud, 1983.

1. The pervasiveness of proportionality in the mathematics curriculum: An overarching concept

Emphasizing and exploiting the pervasiveness of proportionality in the curriculum may help students see mathematics as a coherent discipline grounded in some "big ideas".

Research findings

Conceptual analyses of the notion of proportionality, as well as curricula and textbook analyses, show that:

- Proportionality is associated with a multitude of mathematical concepts and a vast variety of situations wherein proportional reasoning is required.

  Proportionality underlies the development of the idea of rational numbers (fractions, decimals, and percents). For instance, the procedures to find equivalent fractions are very similar to those for finding the missing value in a proportional problem, and comparing two fractions is very similar to solving a ratio comparison problem. In geometry, a very straightforward application of proportionality is that of the size change and similarity of geometrical figures. Proportionality also underlies many important ideas in probability. In a chance game, one can often compare two probabilities by finding the respective ratios of the number of successful outcomes divided by the total number of possible outcomes. And we use proportional reasoning when deciding whether a die or coin is "fair" by comparing the empirically obtained data with a theoretically determined ratio.

  Finally, the proportionality idea underlies other such secondary and higher education topics as linear algebra, the use of linear models in calculus and statistics, and the abstraction in a vector space sense. It is also essential in understanding a variety of problems in physics, chemistry, biology, economics, and so on.

- While proportionality, as such, may receive a lot of curricular attention in the middle school years, the underlying idea of linearity passes through the entire mathematical edifice.

  One of the first (implicit) encounters with proportionality is that of measuring quantities, as this relies on the decision to refer to one quantity as the “unit”, which leads to a linear relation between the physical quantity measured and the number assigned to it. Another one relates to elementary multiplication problems (e.g., If I need 4 handfuls of sand to fill a bucket, how many handfuls do I need to fill 3 buckets?) that also rely on a proportional relation. These evolve in “rule of three” problems (e.g., If I need 10 handfuls of sand to fill 2 buckets, how many handfuls do I need to fill 7 buckets?) in various contexts (cost, sharing, mixtures, and many others) in middle school.

- Proportionality merits more than one or two chapters in middle school math textbooks.

  We argue that proportionality is one of the central “big ideas” in the math curriculum, and that educators can use it to help students see mathematics as an integrated body of interrelated concepts that revolve around a unifying idea.

In the classroom

- It is important to acknowledge proportionality not only as a “big idea” in the math curriculum but also as a connective thread among different mathematics and science topics.
- Students’ prior knowledge and experiences with simpler forms of proportional reasoning (see also below) needs to be taken into consideration in later instruction. Emphasizing and exploiting the pervasiveness of proportionality in the curriculum may help students see mathematics as a coherent discipline grounded in some “big ideas”.

Suggested readings: Behr, Harel, Post, & Lesh, 1992; NCTM, 2000; Vergnaud, 1983; Whitman, 2001.

2.
The importance of mastering one-to-many correspondence

Students' mastery of one-to-many correspondence can be used as a first stepping stone toward understanding proportionality.

Research findings

- Young children, even preschoolers, are capable of making judgments about proportional relations.

- Students at the beginning of primary school use informal strategies to deal with elementary multiplicative problems.

  In the early years of instruction, students use repeated addition to solve elementary multiplication problems, which, as already discussed, are closely related to proportional problems. For instance, they rely on the idea of one-to-many correspondence: If each bucket requires 4 handfuls of sand, one needs 4 + 4 + 4 handfuls of sand to fill 3 buckets. Here, 1 bucket corresponds to 4 handfuls of sand; hence, there is a one-to-many correspondence.

- Older students use informal strategies to solve proportional problems.

  Older students continue to use the idea of one-to-many correspondence, which underlies simple multiplication problems, to solve proportional problems in which the unit ratio is not given. Various terms are used for such informal approaches (e.g., building up, empirical strategies, replications of a composite unit, ...).

  Essentially, these informal approaches come down to adding the values (given in the problem being used) in one or more steps in order to arrive at the desired value. For instance, when one has a recipe for 4 persons and wants to know the recipe for 12 persons, it can be sufficient to reason that one needs the ingredients for 4 persons once more to accommodate 8 persons, and then once more again for 12 persons.

- Adults with little or no formal instruction also use informal strategies, and even well-educated adults will rely on them in daily-life situations.

  Formal instruction in multiplication and division is not needed for people to be able to solve proportional problems. Adults with little or no formal school instruction are able to solve novel proportions problems using values outside the range they typically work with and even, to some extent, in other content domains.

- Informal strategies do not necessarily evolve to more sophisticated ones.

  Students' informal strategies remain very close to the concrete context; that is, they are closely connected to the physical quantities involved in the problem situations rather than the quantitative relations as such. Thus, students typically rely on the scalar relations (i.e., the relation within the same measure space, such as the number of people in the example above) and neglect the functional relations (i.e., the relation between the measure spaces, such as the amount of an ingredient per number of persons).

- Teaching students formal methods does not guarantee that they will use them when appropriate.

  On contextualized problems, secondary school students who were taught the formal method of solving the expression a/b = c/x for the unknown have been found to perform worse than completely illiterate adults who never set foot in school.

In the classroom

- It is important that teachers acknowledge the importance of the informal multiplicative knowledge that students bring to the classroom, make it explicit, and promote its formalization by enhancing students' awareness of their informal knowledge and by offering them ways to represent it.
- Paying serious attention to the contexts of word problems that are used to introduce and practice proportional reasoning and computation may help students to employ their informal knowledge.
- The presence of informal knowledge of one-to-many correspondences seems a powerful thinking schema for the development of multiplicative and proportional reasoning.
- It is important in instruction to support the transformation of the one-to-many correspondence schema into a more powerful one that incorporates the understanding and use of functional relations.

Suggested readings: Nunes & Bryant, 2010; Nunes, Schliemann, & Carraher, 1993; Hart, 1981; Mix, Huttenlocher, & Cohen Levine, 2002.

3. Acknowledging the validity of a variety of strategies

Educators need to understand the variety of strategies that students employ to solve proportional problems.

Research findings

Researchers have identified several approaches that students may employ when solving missing-value proportional problems. We explain all strategies using the following problem: When she makes strawberry jam, my grandmother uses 3 kg of sugar for 6 kg of strawberries. How much sugar does she need for 18 kg of strawberries? The first measure space is the sugar weight; the second, the strawberry weight.

- **Within-strategies**

  These strategies rely on the multiplicative relations within each measure space; that is, they rely on scalar relations. When using a within-strategy, one determines the factor of change within one measure space first, and then applies this factor to the other measure space: We know that instead of 6 kg of strawberries, we now have 18 kg, and we want to know how much sugar we need instead of the 3 kg. The factor of change can be identified by a series of multiplications and/or divisions: One can find out by how much one needs to multiply 6 kg of strawberries to arrive at 18 kg of strawberries (possibly by dividing 18 kg by 6 kg). One needs to multiply the sugar weight by this factor of change as well.

  A simpler variant of the same strategy does not rely on multiplications or divisions, as such, but on repeated addition. This approach relies on the one-to-many correspondence idea and comes down to the building-up approach explained in section 2: For the first 6 kg of strawberries, I need 3 kg of sugar. I also need 3 kg of sugar for the next 6 kg of strawberries, and another 3 kg for the last batch of 6 kg of strawberries. Thus, I need 3 + 3 + 3 = 9 kg of sugar. Such an approach is often used in conjunction with a ratio table that looks as follows:

Strawberries (kg)	6	12	18	24	30	36
Sugar (kg)	3	6	9	12	15	18

- **Between-strategies**
  In this approach, one identifies the multiplicative relation between measure spaces, again either by multiplication or by division: One searches the factor by which one has to multiply (or divide) the strawberry weight in order to obtain the sugar weight, and one applies this factor to the second strawberry weight that is provided. In the above example, we can see that one obtains the weight of sugar by halving the weight of strawberries; so for 18 kg of strawberries, one needs 18/2 = 9 kg of sugar.

  A specific variant of this approach is often called the "unit ratio" approach, sometimes also called the "rule of three". In this approach, one explicitly takes the step to find out the value of the second measure space when the value of the first measure space is 1. In the above example, one first looks for the amount of sugar needed for 1 kg of strawberries (by dividing 3 kg of sugar by 6 kg of strawberries, giving 0.5 kg of sugar per kg of strawberries). One then multiplies this (0.5 kg) by the second value in the first measure space (thus, 0.5 kg of sugar 18 times is 9 kg of sugar).

- **Other strategies**
  Besides approaches in which the reasoning goes within or between the measure spaces, one can also write and formally manipulate the proportion in order to find the missing value. In the original strawberry problem, the proportion is written as 6/3 = 18/x. One can then further solve the problem in two ways. A first method works by creating equivalent fractions (I need to multiply the numerator of the left fraction by 3 to obtain the numerator of the right fraction, so I multiply the denominator of the left fraction by 3, too). A second method is cross-multiplication (I can obtain x by multiplying the numerator of the right fraction [18] by the denominator of the left fraction [3] and dividing it by the numerator of the left fraction [6]).

In the classroom

- To understand students' approaches to proportional problems, it is essential that educators are aware of different strategies.
- All strategies that were described above lead in principle to a correct answer to any proportional missing-value problem. However, it is important that educators are aware that these strategies are very different in nature and therefore have different advantages and disadvantages (described in the following section).

Suggested readings: Noelting, 1980; Karplus, Pulos, & Stage, 1983.

4. Stimulating variety and flexibility in strategy use to develop understanding of proportionality

Students who master a variety of strategies for solving proportional problems may develop a better understanding of proportional situations.

Research findings

- The building-up strategy is less sophisticated, more limited, but more meaningful for the novice learner.

  Among the within-approaches, the building-up approach may appear quite unsophisticated because it does not use the operations of multiplication and division. Some researchers even consider the strong reliance on this approach as a counter-indication of proportional reasoning ability. This approach also depends heavily on the numbers involved in the problem. Indeed, it is not easily applicable in a problem such as: When I need 3 kg of sugar for 6 kg of strawberries, how much sugar do I need for 8 kg of strawberries? Still, when applicable, the building-up approach is a totally appropriate way to solve a proportional problem, and the strategy remains close to the original problem-solving context, with the result that every solution step is meaningful to the problem solver. Moreover, young children (and people with little schooling) can successfully use it.

- Between-strategies are more sophisticated, less limited, but less accessible to the novice learner.

  We can consider between-approaches as truly multiplicative and can use them in situations where the ratio within the measure space makes the application of a building-up approach quite difficult, if not impossible. Moreover, they have the advantage that they are meaningful to the problem solver: The required steps remain close to the original problem-solving context. However, research points out that between-strategies are only rarely accessible to students before they are receive formal instruction.

- Creating equivalent fractions and cross-multiplication are powerful methods but are not transparent to the learner.

  The strategies of creating equivalent fractions and cross-multiplication have as a major advantage that they are algorithmic in nature. One can follow a fixed and guaranteed accurate procedure, which is in principle equally easy for all problems, regardless of the context or the specific numbers involved. These algorithmic approaches are also quite commonly taught in many countries. However, research points out that students themselves rarely choose them, and mistakes are very common. One of the main causes seems to be that these algorithms consist of the blind manipulation of numbers according to formal rules that have no transparent relation whatsoever with the original problem context.

In the classroom

- All strategies are valuable in a curriculum on the concept of proportionality.
- It is important to teach algorithmic methods along with (and preferably after) other methods of solving proportional problems.
- Moreover, it is valuable to make efforts for these algorithmic strategies to become transparent and meaningful for students, so that students do not merely memorize the procedure.
- To become better problem-solvers, students need to understand the relations between the various valid strategies that were determined in research, and gain insight as to when they can apply each strategy most efficiently.

Suggested readings: Karplus, Pulos, & Stage, 1983; Lamon, 2007.

5. Inappropriate additive reasoning is a major source of errors in proportional problems

Before being able to reason multiplicatively, students often approach proportional situations in by focusing on additive instead of multiplicative relations.

Research findings

Section 3 explains that there are various correct strategic approaches for solving proportional problems. In the building-up approach, the multiplicative relations underlying the proportional situation are not directly expressed by multiplicative operations but are accessed through addition and/or subtraction. Still, this approach models the mathematical structure of the situation in a correct manner. However, the use of addition/subtraction is also associated with one of the most frequently reported errors in the proportional reasoning literature: Additive errors occur when students focus on additive rather than multiplicative relations between the given values. They thus subtract one value from another, and apply the difference to the third one. In the strawberry example, students would then notice that when one uses 18 kg of strawberries instead of 6 kg, this means one uses 12 kg of strawberries more. Thus, one needs 12 kg of sugar more, hence 15 kg. In this case, addition is not used as an informal strategy to find a solution to the correctly modeled situation; rather, the problem as such is erroneously modeled in additive instead of multiplicative terms. Given that many informal strategies are additive in nature but also underlie many errors, there is discussion about the extent to which educators should conceive of and teach proportional reasoning as a natural extension of additive reasoning.

- Inappropriate additive reasoning is subject- and task-related. Research identified a number of subject- and task-related factors that influence the occurrence of such additive modelling errors on proportional problems. As an example of the former, this kind of error is more typical for younger children with limited learning experience with the multiplicative relations in proportional situations. But even after instruction, additive errors still occur, particularly on more difficult proportional problems, which brings us to the task-related factors.

  Task-related factors may discourage or enhance additive errors. For example, an important task-related factor associated with fewer additive errors is familiarity with the meaning of the rates (external ratios) involved in the problem (e.g., speed in kilometres per hour, cost in price per unit). On the other hand, a frequently mentioned task-related factor enhancing additive errors is the type of the ratios formed by the numbers in the problem. Specifically, non-integer ratios trigger additive errors. For instance, returning to the original strawberry problem from the introduction: When she makes strawberry jam, my grandmother uses 3.5 kg of sugar for 5 kg of strawberries. How much sugar does she need for 8 kg of strawberries? The ratios within the measure space (8/5) as well as between the measure space (8/3.5) are non-integer. Thus, it is impossible for a student to start using a building-up approach. Also, determining the factor of change by figuring out by how much one has to multiply 5 to obtain 8 requires difficult calculations, as does determining the unit ratio by figuring out how much sugar one needs for 1 kg of strawberries. In those cases, students are frequently reported to revert to erroneous additive responses.

In the classroom

- Additive reasoning can support multiplicative and proportional reasoning. However, instructional overemphasis on additive reasoning can result in students' misapplications. Encountering multiplication merely as repeated addition imposes obstacles on students' reasoning. It would be helpful to instruct students in alternative models of multiplication. One example is to rely on splitting: informal actions like sharing or folding aim at creating multiple versions of an original and, as such, heavily rely on one-to-many correspondence.
- Early explicit instructional attention to the differences between additive and multiplicative reasoning may be useful: at a young age, children can be brought to understand that comparisons and change can be viewed additively as well as multiplicatively. For example, one can describe the age of a 3-year-old and a 6-year-old as the latter being 3 years older, or as being twice as old.

Suggested readings: Confrey, 1994; Hart, 1981; Kaput & West, 1994; Lamon, 2007.

6.
Be cautious of overuses of proportionality

It is important that students are taught not only proportional reasoning strategies but are also taught to distinguish where such strategies can be applied and where not.

Research findings

- Students use proportional reasoning inappropriately.

  Besides the extensive body of evidence of students' reasoning additively in multiplicative situations (documented in section 5), much research has shown that students also apply proportional strategies in situations where this is not appropriate. Especially when problems are presented in a missing-value format, students tend to apply proportional methods, even when such methods do not appropriately model the situation. This is illustrated in various domains of mathematics, including elementary arithmetic, geometry, probability, or algebraic generalization.

  For instance, many students answer "2/6" to the following probabilistic problem formulated in the typical missing-value format: The chance of getting a 6 when rolling a fair die is 1/6. What is the chance of getting at least one 6 when you roll the die twice?

- Additive problems that are superficially similar to proportional problems elicit inappropriate proportional reasoning.

  A particularly persistent problem is the erroneous application of proportional methods to problems with an additive structure. Consider the following problem: Ellen and Kim are running around a track. They run equally fast, but Ellen started later. When Ellen has run 4 laps, Kim has run 8 laps. When Ellen has run 12 laps, how many has Kim run? Many students give proportional answers (in this example: 24 laps) to such problems.

- Inappropriate proportional reasoning is task-related.

  The type of ratio between given numbers also affects students' tendency to use proportional strategies in situations that are not proportional (similarly to the case of inappropriate additive reasoning, see section 5). Consider that students perform better when the problem involves non-integer ratios, such as in the following variant of the previous problem wherein we have replaced the number 8 by the number 6, and the number

12 by the number 10: Ellen and Kim are running around a track. They run equally fast, but Ellen started later. When Ellen has run 4 laps, Kim has run 6 laps. When Ellen has run 10 laps, how many has Kim run?

- Inappropriate proportional reasoning is, to a large extent, instruction-induced.

  While for the runner problem above, students already give inappropriate proportional answers before the start of formal instruction in proportionality, the percentage drastically increases during formal instruction on proportional problems. A major reason for students' strong tendency to apply proportional methods outside their applicability range is the mathematics curriculum wherein, from a certain moment on, teachers pay extensive (and sometimes even almost exclusive) attention to proportionality. Often, this happens with strong focus on the computational aspects of doing proportional problems. Such restricted instructional practices will induce in students an automatic tendency to expect these problems, so that they acquire a kind of "routine expertise" (i.e., the ability to deal with school mathematics tasks quickly and mostly accurately without much understanding) instead of an "adaptive expertise" (i.e., the ability to apply flexibly procedures that are meaningful to them).

In the classroom

- For students to develop adaptive expertise in proportional problems, it is essential that they acquire the habit of explicitly and systematically questioning whether proportionality is the right mathematical model for the situation at hand.
- To this end, it is most useful for educators to provide a variety of proportional and nonproportional problems to be juxtaposed and discussed, also varying the numbers involved.

Suggested readings: Hatano, 2003; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren, De Bock, Janssens, & Verschaffel, 2008; Van Dooren, De Bock, Evers, & Verschaffel, 2009.

7.
Improve teacher knowledge

Teacher education needs to place explicit attention on teachers' development of the content and pedagogical knowledge necessary for effective instruction on proportionality.

Research findings

Research on teachers' understanding of proportionality is limited. Nevertheless, there are clear indications that pre-service and also in-service teachers sometimes struggle with difficulties similar to the ones summarized above.

- Not all teachers are sufficiently flexible regarding strategies for solving proportional problems.

  Research has shown that in-service teachers rely strongly on additive building-up strategies, and have difficulty coordinating two measure spaces multiplicatively in proportional situations. Various studies have also documented that in-service teachers strongly favor the cross-multiplication approach explained above (see section 3) when solving missing-value problems, both in their own solutions and in evaluating students' solutions. Often, they did not acknowledge the value of any of the other strategies explained above, and considered them as less sophisticated or even wrong.

- Not all teachers possess the content knowledge necessary to avoid making additive errors in proportional situations, and the pedagogical knowledge necessary to deal with students' additive errors.

  Pre-service primary and secondary teachers themselves make additive errors when solving proportional problems, such as those about similar shapes. Moreover, many of them (even though solving the problem correctly themselves) may not be able to appropriately explain the origin of students' additive errors. In addition, it appears that teachers may not be prepared to help students who employ additive reasoning in order to correct these problems in a fundamental way, but rather resort to the presentation of procedural methods.

- Many teachers have trouble discriminating between proportional and nonproportional situations.

  Regarding the overuse of proportionality, evidence shows that pre-service teachers struggle with discriminating between situations that are proportional and those that are not. For instance, a large percentage of pre-service teachers provide proportional solutions to problems with additive structure similar to the one presented in section 6.

In the teacher education classroom

- When (pre-service) teachers do not possess the required content (have limited) knowledge to solve proportional problems correctly themselves, they will struggle to come up with helpful tasks and external representations that guide their students toward a thorough understanding and provide them with appropriate feedback in case of difficulties. If this is the case, they will necessarily be more strongly inclined to stick to exercises and representations offered in textbooks and to rely on general and standard feedback when students make errors and experience difficulties, making it difficult for them to implement any of the suggestions mentioned in this guide to deepen students' understanding of proportionality.
- Pre-service teacher education curricula and in-service teacher training need to pay explicit attention to the research findings and educational implications described in this guide.

Suggested readings: Cramer, Post, & Currier, 1993; Ekawati, Lin, & Yang, 2015; Orrill & Brown, 2012; Son, 2013.

8. Concluding educational recommendations

Various relatively easy changes in the curriculum can deepen students' understanding of proportionality.

Bringing together several of the preceding elements leads to recommendations for curricular changes as well as changes in instructional practices. Treat proportionality as a "big idea".

We have suggested that emphasizing proportionality as a connecting thread throughout various topics in the math curriculum may not only enhance students' understanding of proportionality and of the mathematical topics themselves; such measure may also help students to see mathematics as a coherent discipline built around a set of "big ideas".

- Start instruction on proportionality earlier and build on students' informal understandings.

  As discussed above, even before formal schooling on proportionality, students can use informal strategies to approach proportional situations in a correct manner. Moreover, informal strategies employed at a very young age persist for a long time. Therefore, it is recommended to start the teaching of proportionality at a much earlier age than it occurs in typical curricula. Early instruction could build on students' informal knowledge and strategies, and progressively develop them to more abstract strategies, while maintaining a permanent, close link with the more meaningful informal ones.

- Consider proportionality from a modeling perspective.

  It has been suggested that one should consider proportionality from a modeling perspective, in the sense that students will initially deal with concrete proportional situations and build schematic models of these situations, while these schematic models can later become models for proportional situations that students encounter in the future. Models, such as the ratio tables that were illustrated in section 3, play a crucial role in the progression from informal to more formal and abstract knowledge. Teachers can use ratio tables to instruct students at varying levels of understanding, and not only as a tool for computation but also for discussion of the kinds of mathematical models underlying a given situation.

- Avoid overemphasis on the technical aspects of proportional problem solving.

  As we explain above, educators often teach proportionality with a strong focus on the technically correct and fluent execution of certain strategies, with little attention on their applicability in the problem situation at hand. Very often, this comes down to solving series of problems about which it is explicitly stated—or at least implicitly clear—that they are missing-value proportional, or ratio comparison, problems.

- Use nontypical, qualitative problems.

  Several researchers suggest that it is important for students to focus on qualitative aspects of problem situations, such as the quantities involved and the relations between them, before quantifying these. Qualitative problems, which are hardly used in math textbooks, can be used to this end. Consider the problem: Yesterday, grandmother made strawberry jam. Today, she makes strawberry jam using more strawberries but less sugar than yesterday. Will the jam today taste (a) sweeter, (b) less sweet, (c) equally sweet, or (d) is there not enough information to tell? In such problems, students cannot answer using memorized procedures; thus, they require authentic mathematical modelling reasoning.

- Use a variety of tasks.

  Regarding the tendency to overuse proportionality, a larger variation in textbook exercises—beyond the missing-value tasks—seems needed to avoid triggering the proportional scheme merely by using a specific linguistic format. These exercises include classification tasks in which students are not asked to solve a set of given word problems but to group problems that have similar characteristics. Doing so holds students back from blindly applying procedures for which they have been well trained—and may help them look at underlying mathematical models. Other options may relate to problem posing, where students are invited to generate problems (or variants of given problems) themselves.

Suggested readings: Johnson, 2010; Lamon, 2007; Silver, 1994; Streefland, 1985; Van Dooren, De Bock, & Verschaffel, 2010.

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