6

MATHEMATICAL SCIENCES

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Contents

Part One

INTRODUCTION: THE MATHEMATICIANS AND THEIR HERITAGE

(*Q. Mushtaq*)

One distinctive feature of the formation of the early Arab caliphate's culture was its inheritance of heterogenous cultural traditions. The caliphate embraced several centres of ancient Eastern civilizations such as Egypt, Mesopotamia, Iran, Khurasan and Transoxania, many of whose long cultural traditions were directly or indirectly connected with the culture of ancient Greece.

The successors of the Umayyads, the Arab ᶜAbbasid caliphs, were the catalysts in the Muslim cultivation of the arts and sciences, and turned Baghdad into a centre of excellence for the learned and wise within the caliphal lands. Caliphs like Hārūn al-Rashīd (786–809) and al-Ma'mūn (813–33) were keen patrons of the learned at their courts. They made efforts to obtain the best philosophical and scientific texts of ancient Greece and India in Greek, Syriac, Middle Persian and Sanskrit. These were translated into Arabic, sometimes via Syriac, at Baghdad by competent scholars, a process centred on the famous Baytal-Hikma (House of Wisdom) in Baghdad and which, by the time of al-Ma'mūn, had become a well-organized activity of unprecedented scope and vigour. The translation work which began in the second half of the eighth century was practically over by the end of the tenth century, however, never to be taken up again on any significant scale in the Islamic Middle Ages.

A brief look at a few of the translators reveals the variety of their ethnic and religious backgrounds. Some were Persian, like the astrologer Ibn Nāwbakht, who translated from Pahlavi into Arabic. Al-Fazārī, who worked with a scholar from Sind on the translation of the astronomical work, the *Sindhind* (from Sanskrit, 'perfected'), was of Arab descent. The most active translator of medical works in Greek and Syriac, the celebrated Hunayn b. Is'hāq (d. 873), was a Nestorian Christian from Hira. His son and pupil Is'hāq (d. 911), who like his father knew Greek, translated philosophical works of Aristotle, the *Elements* of Euclid and Ptolemy's *Almagest*. Thābit b. Qūrrā (d. 901), a member of the pagan Sabian community at Harran, worked on the translation of mathematical works from Greek.

Some eminent mathematicians

While practically every branch of intellectual thought was pursued during the 'Age of Achievement', the importance attached to the mathematical and astronomical sciences was notable. Among the eminent mathematical scientists who came from or worked in eastern Persia or Central Asia were Muhammad b. Mūsā al-Khwārazmī, who flourished at the ᶜAbbasid court of Baghdad; Abu 'l-Wafā' Muhammad al-Būzajānī, who was patronized by the Buyids; Abū Mahmūd from Khujand; Abū Rayhān al-Birūnī, who flourished under the Ghaznavids; ᶜUmar Khayyām, who became well known under the Seljuqs; Abū ᶜAlī Ibn Sinā; and Nasīr al-Dīn al-Tūsi. Among the others were Abu 'l-ᶜAbbās al-Farghānī, al-Hāzim, al-Karajī, Qutb al-Dīn al-Shirāzī, al-Kāshī, Kamāl al-Dīn al-Fārsī and Abū Sāqr al-Qabīsī.

MUHAMMAD B. MŪSĀ AL-KHWĀRAZMĪ

It is Muhammad b. Mūsā al-Khwārazmī (*fl.* first half of ninth century) who is credited, in his treatise *al-Mukhtasar fī hisāb al-jabr wa 'l-muqābala* [The Condensed Book on Calculations Involving Restoration and Confrontation], with the creation of algebra as we know it in the modern sense.[1] A frequenter of the Bayt al-Hikma established by al-Ma'mūn, he composed the oldest astronomical tables and the oldest astronomical work, which was translated into Latin in medieval Europe. His work focused on lunar anomalies, eclipses, parallaxes, the inclination of the elliptic length of the tropic and on the sidereal year. He was the first scholar in history to discard the idea of the classical conception of the static universe and he strongly upheld the idea of a dynamic universe.

ABU 'L-ᶜABBĀS AL-FARGHĀNĪ

Abu 'l-ᶜAbbās al-Farghānī (d. 860), known as Alfraganus in the West, supervised the erection of a Nilometer at al-Fustāt in Egypt and measured the diameter of the earth and other planets. He also accurately calculated the distances between the planets. His *jawāmiᶜᶜilm al-nujūm* [Compendium of astronomy] was highly valued and remained in use throughout Europe for several centuries, most of his works being transmitted to Europe through Latin and Hebrew.

ABU 'L-WAFĀ' AL-BŪZĀJANĪ

It was Abu 'l-Wafā' Muhammad al-Būzajānī (940–98), from Khurasan, who gave the world the first known tables of tangents calculated for every 15'. He was the first to show the generality of the sine theorem relative to triangles. He and other mathematicians formulated and successfully developed a branch of geometry which dealt with problems leading to algebraic equations of the third and higher degrees. This correlation of geometry with algebra, and the geometric method of solving algebraic equations, anticipated Descartes' discovery of analytical geometry in the seventeenth century (see further, Part Two below).

ABŪ RAYHĀN AL-BĪRŪNĪ

The Khwarazmian Abū Rayhān Muhammad al-Birūnī (973–1048) introduced the idea of a 'function' which describes the correspondence of two numbers and the dependence of

[1] Al-Khwārazmī, 1939.

one number on the other, a concept which has become one of the most important ideas in mathematics. He made an accurate determination of specific gravities. In his *al-Qānūn al-Mas^{c}ūdī* [Canon of (Sultan) Mas^{c}ūd], he discussed for the first time the question of whether the earth rotates around its axis and gave the true explanation that the rising and setting of the heavenly bodies is a result of the rotation of the earth; he thus pointed to the error in the geocentric conception of the solar system.

ABŪ ^{c}ALĪ IBN SĪNĀ

Abū ^{c}Alī Ibn Sīnā (c. 980–1037), known to the West as Avicenna, was not only a great physician, a philosopher and a philologist, but also a mathematician. He devoted four books of his *Kitāb al-Shifā'* [Book of Healing] to the mathematical sciences, thus confirming the mathematical orientation that had characterized Hellenistic-Islamic philosophy from its beginning. Unlike his predecessors, al-Kīndī and al-Farābī, Ibn Sīnā no longer conceived of mathematics as an activity in some way isolated from philosophy but rather as an integral part of a philosophical synthesis. He renounced the traditional language of arithmetic and took up that of the algebraists to explain the successive powers of an integer. He took up the theorem of Thābit b. Qūrrā on amicable numbers and several problems on congruences. In his *Shifā'* he gave the example of the first case of Fermat's conjecture, also treated by at least two other mathematicians of the tenth century, al-Khujandī and al-Khāzin. Ibn Sīnā also wrote on whole or fractional, rational or algebraic irrational numbers. He gave a system of classification of arithmetic, *hisāb* (calculation) and algebra different from the Graeco-Hellenistic one. He threw new light on ontology and on logic, considering the latter as the science of truth concerning the studies of propositional forms and the process of reasoning. He brought a significant improvement to all parts of Aristotle's logic, and distinguished between conjunctive-conditionals and disjunctive-conditionals. He made a great contribution to the theory of propositions. The legacy of Ibn Sīnā is considerable. The Persian scholar Sa^{c}īd Nafisī calculated that he wrote 456 works in Arabic and 23 in Persian, but those genuinely attributable to him must be less.[2] According to the library catalogues of different countries, 160 of his works are preserved today.

^{c}UMAR KHAYYĀM

In the Seljuq period, ^{c}Umar (or Omar) Khayyām of Nishapur (*c.* 1048–1123) and ^{c}Abd al-Rahmān al-Khāinī were the leading scientists who conducted astronomical observations. Their works resulted in the adaptation of a new era, known as the *jalālī* era. So perfect was

[2] Asimov, 1986, pp. 220–43.

this work that there is an error of only one day in 5,000 years. [c]Umar Khayyām was at the same time a great mathematician and a distinguished philosopher and astronomer; whilst a philosopher, he was a follower of Ibn Sīnā, whom he interpreted with keen rationalism. His *Algebra* contains equations of the third degree. Like his Arab predecessors, he provided both arithmetic and geometric solutions for quadratic equations. For cubic equations, he was the first to classify them systematically and to obtain a root as the abscissa of a point of intersection of a circle and a rectangular parabola, or of two rectangular hyperbolas. He was also the first to solve geometrically every type of cubic equation that possesses a positive root, and was aware of the trend of finding algebraic solutions of the cubic equation, whose solution in its generality was found only in the sixteenth century.

[c]Umar Khayyām seems actually to have studied numerical solutions, specifically of equations of the form $x^n = a$, (where n is a positive integer). The case for $n = 2$ was also known to Euclid, but any evidence of the generalization of the law for other values of$n$ first appears, it would seem, in [c]Umar Khayyām's *Algebra*. He did not give the law, but he asserted that he could find the 4th, 5th, 6th and higher roots of numbers by a law which he had discovered and which did not depend upon geometric figures and that he had set this forth in another work, now lost. Furthermore, he was the first to mention the later Italian mathematician Cardano's 13 forms of the cubic which have positive roots. He was the originator of some of the basic ideas underlying what is now known as non-Euclidian theory of parallel lines, at least so far as it antedates the work of Girolamo Saccheri, and his suggested theorems and proofs of Euclid's fifth postulate are essentially the same as the first few of Saccheri's (see further in Part Two below).

NASĪR AL-DĪN AL-TŪSĪ

Nasīr al-Dīn al-Tūsī (d. 1274), much influenced by [c]Umar Khayyām's work, continued efforts to prove the parallel postulate. His writings influenced Saccheri's work on non-Euclidean geometry, *Euclides ab omni naevo rindicatus* (1733), which is generallycon-sidered as the first step in a non-Euclidean geometry. Al-Tūsī's views also influenced John Wallis (seventeenth century), who translated his works into Latin, published them and used them in his famous Geometrical Lectures at Oxford.

AHMAD AL-ŪQLĪDISĪ AND JAMSHĪD AL-KĀSHĪ

The Muslims also made an important contribution to the history of decimal fractions. The first writer known to have used decimal fractions was the Syrian Ahmad b. Ibrāhim al-Ūqlīdisī (*fl.* mid-tenth century).[3] All that is known about the author is his *Kitāb al-Fusūl fi 'l-hisāb al-hindī* [Book of Chapters on the Indian Numerals], in which he uses decimal fractions, appreciates the importance of a decimal sign, and suggests one, that is, the point /./. The next writer known to have used decimal fractions was the Khurasanian al-Nasāwī, who flourished in the first half of the eleventh century. (On the continued use of the sexagesimal system, see Part Two, below.)

In Mongol times the intellectual life of Central Asia and eastern Persia suffered a regression. There was, however, a fresh spurt of intellectual activity under the Timurids, who were in power from about the middle of the fourteenth century until the beginning of the sixteenth century. It was during the reign of Timur's grandson Ulugh Beg (1394–1449) that the study of astronomy especially reached new heights at Samarkand and that the noted Persian mathematician Jamshīd b. Mas[c]ūd al-Kāshī (d. 1429) flourished under his patronage. Ulugh Beg founded in his capital an imposing observatory, which made this ancient Central Asian city a noted centre for astronomy. He was also an astronomer in his own right, being one of the first to advocate and build permanently mounted astronomical instruments. His catalogue of 1,018 stars, written in 1437, was the only such undertaking carried out between the time of Claudius Ptolemy (*fl.* mid-second century) and Tycho Brahé (d. 1601).[4] In this new catalogue, the positions were given to the nearest minute of arc, and attained a high degree of precision for the period.

Jamshīd al-Kāshī wrote at Samarkand his *Miftāh al-hisāb* [The Key to Arithmetic], a comprehensive, clearly written and well-arranged handbook for merchants, clerks, surveyors and theoretical astronomers which is also an important work in the history ofnumbers, especially for its full and systematic investigation of decimal fractions. Al-Kāshī built up the scale of decimal fractions by analogy with that of the sexagesimals. He realized the importance of these fractions more than his predecessor the Syrian al-Ūqlīdisī had done, claiming them as his own invention and giving them a special name, *al-kusūral-a[c]shāriyya*. Muslim mathematicians, who had been using sexagesimals in their large-scale computations, began to use decimals after al-Kāshī's time. Through the use of decimals, al-Kāshī's approximation of π was more accurate than any of the values given by his predecessors. He was also the first to solve the binomial later known as Newton's, with its solution in this same treatise on arithmetic, and he further wrote *al-Risāla al-muhītiyya* [The All-Embracing Treatise on the Circumference], based on the sexagesimal system.

Nevertheless, the Central Asian cities never quite regained their pre-Mongol intellectual activity and excellence. This is related to the decline of the Muslims as a whole in scientific

3 Saidan, 1966, p. 475.
4 Krisciunas, 1992, pp. 3–6.

intellectual development. One factor was the decline of curiosity on the part of individuals, and here the victory of the Ash^{c}arites may have had serious consequences for Muslim civilization since it led to the development of *taqlīd* (the unthinking acceptance of tradition in religion and other spheres), which was the antithesis of *ijtihād* (the exertion of effort in resolving religious and legal problems), a process which now became regarded as closed. Hence learned scholars were gradually forced to concede that it was no longer allowed for anyone to exercise independent reasoning in matters of religion.

Part Two

THE MATHEMATICAL SCIENCES

(*J. L Berggren*)

Mathematics in medieval Islam was international in its scope and in its intended audience. Its mathematicians drew on sources that ranged widely in time and space, and both eastern and western parts of the medieval Islamic world made important contributions to the various divisions of the mathematical sciences. Thus this brief history will of necessity trespass into other regions in its account of the conditions, consequences and achievements of Central Asian mathematics.

Arithmetic

The earliest treatise on arithmetic known to us from the Islamic world is that of Abū Ja^{c}far Muhammad b. Mūsā al-Khwārazmī. He worked in Baghdad in the first half of the ninth century, but his ethnic name Khwārazmī points to at least his ancestors' origin in the region between the lower courses of the Amu Darya (Oxus) and Syr Darya (Jaxartes) rivers. His work is only extant in a Latin translation from the twelfth century,[5] and, indeed, it was the earliest Arabic arithmetic to appear in Latin. Because the work introduced the Hindu

[5] See Sezgin, 1974, p. 238; one suggestion has been that it was called the *Kitāb al-Jam^{c} wa 'l-tafrīqbi-hisāb al-Hind* [Book of Addition and Substraction According to the Hindu System of Calculation]. Until recently our knowledge of this treatise has been confined to one manuscript, in Cambridge, United Kingdom, but now a second manuscript has been found in New York, one which differs from that in Cambridge in several ways.

decimal positional system to both the Islamic and Latin worlds, it has, as A. Yuschkevitch observes, ‘not only for mathematics, but for the whole cultural development of the world … a great significance’.[6] And, via its Latin offspring, al-Khwārazmī’s work has left a lasting impression on those Western languages in which the word for any systematic method of computing (as in the English ‘algorithm’) is derived from the Latin form of al-Khwārazmī’s name, ‘algorismi’.

The importance of al-Khwārazmī’s book lies in its being the first of a series of works developing the base ten positional system which the Islamic world had inherited from the Indians. The end product of the tradition that al-Khwārazmī’s work originated may be seen in the work of Jamshīd al-Kāshī (see Part One, above), whose *Miftāh al-hisāb* [The Key to Arithmetic] treats of arithmetic, algebra and mensuration, and gives instruction on the extraction of roots of arbitrary orders, the use of the table of binomial coefficients[7] (known today as Pascal’s triangle) and decimal fractions.

All of these developments, however, took place after al-Khwārazmī. Thus decimal fractions appear in the first extant work of Arabic arithmetic, that of al-Ūqlīdisī in the tenth century (see Part One, above), and in the work of al-Samaw’al b. Yahyā in the twelfth century. But it is not known what any of these writers owed to the other, and al-Kāshī’s claim to have invented decimal fractions must be taken as representing his honest belief. Perhaps[8] al-Kāshī’s familiarity with the Chinese astronomers’ system of measuring time in days and ten-thousandths of a day (called *fên*) inspired his invention of decimal fractions, which he used, among other purposes, for displaying the results of his calculation of π to 16 decimal places.[9]

Al-Kāshī’s *Miftāh* was held in such high regard that the Persian scholar Muhammad Tāhir Tabrisī informs us that for two centuries after its composition it remained the standard arithmetic text in Persian *madrasas* (colleges for higher religious studies). And in a preface to his astronomical tables, written some eight years after al-Kāshī’s death, Ulugh Beg, to whom al-Kāshī dedicated this work, refers to him as ‘the admirable master, known among the famous of this world, who had mastered and completed the sciences of the ancients and who could solve the most difficult questions’.

However, the works both of al-Khwārazmī and of al-Kāshī illustrate the fact that by no means all arithmetic in the Islamic world was based on the decimal system, for both also treat the sexagesimal system, a positional system based on 60 rather than 10. And although

[6] Yuschkevitch (quoted in Sezgin, 1974, p. 238).
[7] It appears that this was first discovered by al-Karajī, *c.* A.D. 1000.
[8] See Kennedy, 1964.
[9] What was remarkable was not only the number of places but the fact that al-Kāshī was able to control the round-off errors in the calculation so that he knew the accuracy of the results of his calculations.

decimal fractions were an Islamic contribution, sexagesimal fractions had been used since at least the second millennium B.C. Knowledge of the sexagesimal system may have come to medieval Islam through Greek or Sanskrit astronomical works,[10] and a standard name for the system in the Islamic world was 'the astronomers' arithmetic'.[11] A systematicp-resentation of the system is found in one section of the *Usūl hisāb al-Hind* [Principles of Hindu Reckoning] by Kūshyār b. Labbān al-Jīlī (*fl.* second half of tenth century).[12] However, despite the existence of sexagesimal multiplication tables,[13] multiplication and division in the sexagesimal system were often accomplished by converting the numbers to a decimal representation, performing the operations there, and then converting the answers back to the base 60, a procedure referred to as 'levelling'.

The use of the base 60 for dealing with fractions, found in the Islamic world as early as al-Khwārazmī, also occurs in the work of Abu 'l-Wafā' al-Būzajānī. In his *Kitābal-Manāzil fīmā yahtāju ilayhi al-kuttāb wa 'l-ᶜummāl min ᶜilm al-hisāb* [Book of the Stages Concerning What Secretaries and Financial Officials Need in the Way of Arithmetic], Abu 'l-Wafā' tells how to use the base 60 to deal with fractions. It has in fact survived until today in our writing that a certain angle is, e.g., $127\,^{\circ}30\,'41''$,[14] According to A. S. Saidan, the base 60 served in commercial computations much the same purpose as our percentages.[15]

With al-Būzajānī's work we come to the third major system of arithmetic in the Islamic world, that of finger reckoning. In medieval Islam this was also known as the 'system of the Arabs and the Byzantines'. However that may be, the Roman biographer Plutarch (46–127) reports that a system of finger reckoning was known to the Persians, and the fourteenth-century Persian historian and geographer Hamdallāh Mustawfī credits Ibn Sīnā with the invention of a system of calculation by this method. These data suggest that this system had many variants, but the general principle of all variants was that the numerals were represented by bending the fingers into certain standard positions in order to retain the results of intermediate stages in mental calculations. Unlike the Hindu system, whose operations were performed with a finger or stylus on a dust board or (later) on paper with

[10] The system was widely used in Greek astronomical texts known in the Islamic cultural domains, but al-Khwārazmī in his *Usūl hisāb al-Hind* attributes it to the Hindus.

[11] It is so described in the *Miftāh* of al-Kāshī, who devotes *Maqāla* III of that work to the topic.

[12] The circulation of texts and ideas around Asia is illustrated by the fact that his *zīj* (astronomical handbook), the *Madkhal ilā ᶜilm al-nujūm*, extant in both Arabic and Persian, was early translated into Chinese.

[13] Referred to, but not present, in Kūshyār's work. See also King, 1974*a*; 1974*b*; 1979.

[14] A similar use of sexagesimals for representing fractions is found in the portion of Kūshyār's work dealing with decimal arithmetic.

[15] Saidan, 1974, pp. 364–75.

a pen, the fact that operations in the system of finger arithmetic were performed mentally led to considerable attention being paid to computational short cuts.[16]

Al-Būzajānī may have addressed his work to secretaries and financial officials, but computation was also used by scientists,[17] and noteworthy examples of scientific computing are the sophisticated algorithms for solving equations or for computing corrections to a quantity which is initially only calculated very roughly. Two examples of the former associated with Central Asian mathematicians are the method of Habash al-Hāsib for the iterative solution of $t = \theta - m\dot{}\sin\theta$ for $\theta = \theta(t)$, an equation known today as Kepler's equation,[18] and al-Kāshī's iterative method for solving cubic equations arising from the problem of trisecting angles.[19] An example of the latter is the method of calculating the moment of true conjunction of the sun and moon, starting with a mean conjunction of these two luminaries. E. S. Kennedy describes such a method used in the Chinese-Uighur calendar and found in the *Zīj-i Khāqānī* [Royal Astronomical Tables] of al-Kāshī.[20]

A notable development of the iterative techniques for solving equations is found in the works of Sharaf al-Dīn al-Tūsī (d. *c*. 1213), whose *On Equations* gives not only methods based on numerical tableaux for solving cubic equations, but also arguments for the validity of these methods.[21] Al-Tūsī's mastery of both ancient mathematics and that of his own time allowed him to derive conditions for the solvability of cubic equations which we would most naturally verify today by means of the differential calculus but for which he probably used a sophisticated mastery of Euclidean geometric algebra.[22]

Of course, by ' arithmetic' the Greeks meant the theory of numbers, something theArabic writers called either 'the science of numbers' (a direct translation of the Greek) or (e.g. al-Fārābī) 'the theoretical science of numbers'. The Islamic tradition in this area[23] was based on the number-theoretic books of Euclid's *Elements* (Books VII–IX) and that treasury of Pythagorean number lore, Nicomachus' *Introduction to Arithmetic*, one of the earliest Islamic contributions to number theory was to a favourite topic of Nicomachus. This was Thābit b. Qurra's discovery, in the late ninth century, of a condition for each of two numbers to be the sum of the proper divisors of the other.[24] Although Thābit's

16 See, for example, Saidan, 1974, p. 372.
17 Abu 'l-Wafā' for example, was an astronomer, and suggested taking the radius in computing the Sine function equal to 1.
18 See Kennedy et al., 1983, pp. 513–16.
19 See Aaboe, 1954, pp. 24–9.
20 Kennedy, 1964, pp. 435–43.
21 See Rashed, 1985.
22 See Hogendijk, 1989, pp. 69–85; Berggren, 1990, pp. 304–9.
23 See the survey of Islamic work in number theory in Naini, 1982.
24 A pair of such numbers, e.g. 284 and 220, the Greeks called 'amicable'.

theorem was not easy to apply, Kamāl al-Dīn al-Fārisī in the thirteenth century was able to find a new pair of amicable numbers (17,296 and 18,416), a pair rediscovered by Pierre de Fermat in the seventeenth century. Al-Fārisī's work was conducted in the context of a systematic study of the sum of divisors of a number.

At the same time, there was a strong tradition of Diophantine analysis, carried on both in algebra and in number-theoretical investigations of questions which would arise on reading Diophantus' *Arithmetics*. An example of this latter is the proof of Abū Ja^c^far al-Khāzinal-Khūrasānī (d. *c*. 965) that, given a whole number a, the equations $x^2 + a = m^2$ and$x^2 - a = n^2$ are simultaneously solvable for whole numbers x, m and n if and only if a is twice the product of two whole numbers whose squares add up to a square. In such a case we may take x^2 to be that square. According to him, Abū Mahmūd Khujandī (from Khujand, in Transoxania) gave an incorrect proof of the impossibility of solving the first case of what was to become Fermat's conjecture: for $n > 2$ there are no whole-number solutions to $x^n + y^n = z^n$, by no means the last incorrect proof associated with Fermat's conjecture! In the twelfth century Ibn al-Khawwām stated the same for the next case, namely $x^4 + y^4 = z^4$,but neither he nor his commentator Kamāl al-Dīn al-Fārisī proved it.

Finally, on the subject of arithmetic, one measure of the progress made in the conception of numbers which occurred from the ninth century onwards is the arithmetic treatment of Euclid's Book X, a work devoted to a classification of quadratic irrationals considered as geometric magnitudes. Beginning with Abū ^c^Abdallāh Muhammad al-Māhānī (*fl.* mid-ninth century), one finds developing an Islamic tradition of treating irrational magnitudes as irrational numbers. This arithmetic treatment of Book X leads to the point where 'Euclid's propositions are directly given as collected rules of operations on number irrationalities.'[25] (For the development of the idea of ratio as being a number, see the remarks on ^c^Umar Khayyām below)

Algebra

Algebra was closely connected with arithmetic in the Islamic world, for as A. I. Sabra[26] has pointed out, both were studies whose object was to compute from one or more given numbers an unknown number (whether it be the sum or product of two given numbers or the root of a quadratic equation with given coefficients). Indeed, many of the arithmetic books contained chapters on algebra and many of the works on algebra had the word *hisāb* (calculation) in their titles. Moreover, as with arithmetic, algebra was another ancient area

[25] Matvievskaya, 1987, p. 272.
[26] Sabra, 1971.

of endeavour which Islamic mathematicians systematized. Finally, as with arithmetic, the first known books in algebra stem from writers connected with Central Asia.

One of these writers we have already met: Muhammad b. Mūsā al-Khwārazmī, who dedicated his *Algebra* to the caliph al-Ma'mūn.[27] The first part of this tripartite work is a basic introduction to the algebra of equations of at most the second degree. The author prescribes the procedures for solving each of the six types into which he classifies such equations, presents examples, and gives demonstrations of the validity of the methods. In these demonstrations he uses informal geometric arguments which are ultimately of Babylonian origin.[28] The *Algebra* was twice translated into Latin in the twelfth century, by Robert of Chester and then by Gerard of Cremona, and in that form made a great impact on the West.[29] Another early writer on algebra was ᶜAbd al-Hamīd Ibn Turk, whose origins lay either in Khuttal or in Gilan and who was apparently a contemporary of al-Khwārazmī. His book *al-Darūrāt fi 'l-muqtaranāt* [The Logical Necessities of Mixed Equations] is extant only in part.[30] That it is not possible to assign priority to one or the other does not matter very much, if one accepts E Sezgin's quite reasonable view that both works reflect activity that was already going on when their authors entered the scene.[31]

Following upon these two writers, Abū Kāmil Shujāᶜ b. Aslam of Egypt (*fl.* second half of ninth century), in his *Kitāb fi 'l-Jabr wa 'l-muqābala* [Book on Algebra], extended the algorithms which al-Khwārazmī and Ibn Turk had stated for operating with polynomials to powers as high as the eighth, and developed the arithmetic of binomial expressions, which Euclid had treated geometrically in Book X of his *Elements*. About a century later, Abū Bakr al-Karajī (*fl. c.* 1000), in his *al-Fakhrī* [The Splendid (Book)], became the first person to state general rules allowing the user to operate with polynomials of arbitrary degree. However, since he had no algebraic symbolism, al-Karajī relied on rules for manipulating coefficients of polynomials arranged in the columns of a table. With this technique he was able, for example, not only to multiply but, in some cases, to divide such polynomials as well. It appears however that, not knowing the rule $-a-(-b) = -(a-b)$, he was unable to master all cases of division of polynomials, and we first find this done successfully in the writings of al-Samaw'al b. Yahyā, a Jewish convert to Islam who died in 1174 at Maragha in Azerbaijan. He did systematic work with decimal fractions in problems dealing with

[27] The full title is *al-Mukhtasar fī hisāb al-jabr wa 'l-muqābala* (see Part One above). Text and translation in Rosen, 1831.
[28] Gandz, 1936, pp. 523–4; Høyrup, 1986, pp. 445–84.
[29] Toomer, 1973, p. 362, calls it 'the chief influence on medieval European algebra'.
[30] Sezgin, 1974, p. 241; Sayili, 1962, pp. 87–91, who has published the extant part, points out that the date of Ibn Turk can be approximated only from the date of death, 910, of a man thought to be his grandson.
[31] Sezgin, 1974, p. 241.

root extraction and the approximation of roots of polynomials. Here one should highlight his statement, in his work, *al-Tabsira fī* ^{c}*ilm al-hisāb* [The Perspicacious (Book) onArithmetic], both of the intricate cases of the law of signs and of the general law of integer exponents, $a^{m}a^{n} = a^{m+n}$, valid for negative integers *m* and *n* as well as positive.

Moreover, al-Samaw'al was able to generalize algorithms for extracting square roots of ordinary whole numbers to rules for square roots of polynomials as well. In accomplishing these tasks, he not only shows his awareness of the potentially infinite nature of the processes (by referring to a finite part of the answer as 'the answer obtained so far') but also gives a recursive rule for writing down the coefficients of all the remaining powers.

A different approach to algebraic problems is that of ^{c}Umar Khayyām. In his *Maqāla fi 'l-jabr wa 'l-muqābala* [Discourse on Algebra], which he dedicated to Abū Tāhir, the chief *qādī* (judge) of Samarkand, ^{c}Umar Khayyām classifies the polynomial equations in a single variable of degree at most 3 according to the number of terms involved and then discusses each case where there is a positive real solution. His book follows the Central Asian tradition set by al-Khwārazmī of being entirely rhetorical, lacking the algebraic symbolism which developed in the Muslim West. ^{c}Umar Khayyām, moreover, avoids subtracted quantities in his classification of different types of equations, so that in the end he has 25 species of equations. His methods of solution, as he says in the preface to his work, use Euclid's *Elements* and *Data* and Apollonius' *Conics*. Terms such as x^3 or ax^2 he interprets geometrically as volumes, and his solutions are represented as line segments.

However, in ^{c}Umar Khayyām's view the fact that his approach is geometric does not mean his work is not algebra. Indeed, he says in the introduction that, 'One of the branches of knowledge ... is the science of algebra, which aims at the determination of numerical and geometrical unknowns.' The important criterion was evidently that one was searching for an unknown quantity, and that in doing so one was using rules and procedures which had been used by algebraists from the time of al-Khwārazmī and Ibn Turk. The fact that the quantity one was searching for was a geometric magnitude and that one also used some theorems of geometry in no way made the work less algebraic. Thus algebra was a branch of mathematics with very wide applications. On the one hand, it could contain many elements, and solve many problems, which we would think of as geometric. On the other, it was also considered a fundamental part of *hisāh*, the science of finding unknown quantities from known quantities.[32]

In his study of cubic equations, ^{c}Umar Khayyām was aware that he was building on work of al-Khāzin. Earlier, al-Māhānī had shown that Archimedes' problem of dividing a

[32] For example, al-Kāshī in his *Miftāh* devotes *Maqāla V* to the subject of 'finding unknowns by algebra and the rule of double false position and other methods of calculation'.

sphere could be stated in terms of a cubic equation, and ᶜUmar Khayyām informs us that Abū Jaᶜfar wrote a treatise containing the solution by intersecting conies.

Trigonometry

One of the chief contributions of Islamic mathematics was the development of plane and spherical trigonometry.[33] Although mathematicians from many parts of the Islamic world contributed to this endeavour, some of the most important applications were made by the mathematicians of Central Asia in the context of astronomical research. One of the most eminent of the early practitioners of mathematical astronomy was Habash al-Hāsib, originally of Merv but largely resident in Baghdad (*fl.* ninth century), who, as far as we know, was the first to calculate tables of auxiliary functions. These are combinations of trigonometric functions which are of little interest in themselves but which often appear in formulae of spherical astronomy, such as those in astronomical timekeeping.[34] They are thus of great service in computing astronomical tables. Perceiving the utility of such aids testifies to Habash's insight into the structure of a variety of seemingly different mathematical expressions.

Astronomer-mathematicians in the Islamic world spent considerable time improving the tables of the trigonometric functions that they had inherited from the Hindus. The climax of this development was the work of Ulugh Beg (see Part One, above), who composed*c.* 1440 his sine tables for each minute of arc to 5 sexagesimal places, an accuracy of almost 1 part in 1,000 million. This accuracy was, of course, based on the iterative method al-Kāshī used for computing Sin (1 °) from the value of Sin (3 °), which we mentioned above.

It appears that the development of spherical trigonometry took place during the latter half of the tenth century. The results of this work, as well as something of its history, are recounted by al-Birūnī in his *Kitāb Maqālīd ᶜilm al-hay'a* [Book of the Keys to Astronomy], which he evidently wrote at the request of the Khwarazmian ruler Abū Nasr b. ᶜlrāq. From this account, it appears that al-Būzajānī played a major role in the history of spherical trigonometry, for he explained the ubiquitous Rule of Four Quantities, the Law of Sines and the Law of Tangents.[35] It was Nasīr al-Dīn al-Tūsī who, at Maragha in the thirteenth century, completed and summarized trigonometry in his *Kitāb al-Shakl al-qattāᶜ* [Book on

[33] The latter is the trigonometry of triangles on the surface of a sphere whose sides are arcs of great circles on that sphere.
[34] For a survey of medieval literature on this topic, see King, 1990, pp. 27–32.
[35] See Debarnot, 1985, for the full text and French translation. An abridged account of this very interesting story may be found in Berggren, 1987, pp. 16–17.

the Complete Quadrilateral]. It is this work which marked the emergence of trigonometry as a discipline independent of astronomy, to which it had been linked for so long.[36]

A point which should be mentioned here, and which also applies to the subjects discussed earlier, is that each of these mathematical sciences acquired Islamic dimensions as its practitioners became aware that their disciplines could be used to provide exact solutions to problems unique to Islamic societies. In the case of arithmetic, such problems include the calculation of *zakāt* (alms-tax), as well as the seemingly endless calculations necessary to obtain tables of the times and direction of prayer. In the case of algebra, as we find from al-Khwārazmī's work, it includes the division of legacies (the so-called *ᶜilm al-farā'id*, or science of obligatory shares). In the case of geometry, it was the application of that subject to the measurement of surfaces and volumes arising from the features of many mosques.[37]

Trigonometry found several areas of application, one of these being the determination of the direction of prayer, i.e. the determination of the direction of Mecca (the *qibla*), for a given locality. One of the masters of the application of spherical trigonometry to the basic problems of mathematical geography, which would permit the solution of the *qibla* problem, was al-Bīrūnī.[38] Suffice it to mention here that the goal of this work was to put Ghazna (in what is now eastern Afghanistan) 'on the map' by determining its latitude and longitude and from them its *qibla*. The sophistication of this tradition of determining the *qibla* is indicated by the recent appearance, on the modern antiquities market, of a device consisting of a circular disk on which is a co-ordinate net so devised that, when a ruler is rotated around the centre to pass through the cell bearing the name of some city, the end of the ruler indicates on the scale of the outer rim the *qibla* of that city. The scale on the ruler shows the distance between Mecca and that city. Although this particular device was made in Isfahan *c*. 1700, there is every reason to believe that it reflects a medieval tradition.

Geometry

Following the translation of many of the major and minor works of the Greek geometers into Arabic, the geometers of medieval Islam, and of Central Asia in particular, extended the frontiers of geometric research and opened up whole new areas as well.

Unlike other branches of the mathematical sciences, geometry had come to Islam with a logical structure, based on definitions, axioms and postulates. This situation, not

[36] A similar development did not take place in the West until Regiomontanus completed his *De triangulis omnimodibus* in 1464.
[37] See Dold-Samplonius, 1992, pp. 193–242.
[38] Details can be found in his *Tahdīd nihāyāt al-amākin* [Determination of the Co-ordinates of Localities]. See Kennedy, 1973; Ali, 1967.

surprisingly, attracted the attention of the geometers of the Islamic world to foundationalquestions in the subject. A prominent example of this is research into the question of Postulate 5 of Book I of Euclid's *Elements*, the so-called parallel postulate. Since this work has been well investigated by B. A. Rosenfeld, and the Arabic texts have been made available by K. Jaouiche, we shall merely state here that the tradition of the quadrilateral with two right angles, found in the works of ᶜUmar Khayyām and then in Nasīr al-Dīn al-Tūsī, became known in Europe through the pseudo-al-Tūsī recension of Euclid's *Elements* which was printed in Arabic in Rome.[39]

Another basic question in geometry was, as already indicated, that of the foundations of the theory of proportion, i.e. of when two ratios should be regarded as being the same.[40] As in so many other areas, a major contribution to the question was made by ᶜUmar Khayyām, who argued in his work on the difficulties in Euclid[41] that Definition 5 of *Elements* V hid the true nature of proportion and should be replaced by one based on the idea of anthyphairesis. This procedure, based on successive subtraction, is hinted at in Aristotle, and is used in the Euclidean algorithm (*Elements* VII.2) to find the greatest common divisor of two whole numbers. ᶜUmar Khayyām not only proved the equivalence of the two definitions of ratio, but also came to a general conception of real number in his notion that any ratio could be treated as a kind of number even though, strictly speaking, it was not a number.

By no means all geometric work was devoted to foundational questions, however. Particularly noteworthy are the geometers, many of Central Asian origin, who assembled at the Buyid court of ᶜAdud al-Dawla and his successors in southern Persia in the late tenth century. Among them was Abū Sahl al-Kūhī of Tabaristan (*fl.* second half of tenth century), called 'Master of His Age in the Art of Geometry' by his two younger contemporaries, Abu '1-Jūd and al-Shannī.[42] Among al-Kūhī's writings are works on the regular heptagon and on duplicating the cube and trisecting the angle; correspondence about his remarkable new discoveries in geometric mechanics;[43] a study of geometric problems suggested by the problem of constructing an astrolabe; and finding the volume of a paraboloid of revolution. For the astrolabe, he also invented a new method for solving the uniquely Islamic problem of drawing the projection of the azimuth circles on that instrument.[44] A colleague of al-Kūhī's was Abū Hamid Ahmad al-Saghānī (called 'The Astrolabist'; *fl.* tenth

[39] According to one suggestion, it may be the son of al-Tūsī, namely Sadr al-Dīn, who composed this.
[40] See Plooij, 1950.
[41] See Amir-Moez, 1959, pp. 276–303.
[42] Quoted in Hogendijk, 1985, p. 113.
[43] Published in Berggren, 1983.
[44] Published in Berggren, 1982.

century), whose name indicates an origin on the upper Oxus. He was the author of a study of projection of a sphere[45] on to a plane perpendicular to its axis from a point on the axis but not on the sphere, a generalization of stereographic projection which gave rise to a variety of interesting mathematical problems and curious astrolabes. Both of the above mathematicians participated in a lively controversy at the Buyid court on the following questions: the admissibility of a construction of the regular heptagon in a circle thatArabic sources attribute to Archimedes; and the validity of constructions proposed by various tenth-century geometers.[46] Of course, the whole debate took place within the context of active work on the other famous problems of Greek antiquity, such as the trisection of the angle and the duplication of the cube.

We must close with a mention of some of the applications of geometry that were realized by the mathematicians of Central Asia. Pride of place in this group must go to al-Birūnī, whose geometric methods in cartography, geodesy and astronomy still excite admiration. We have already mentioned his *Tahdīd nihāyāt al-amākin*, where he solves the problems of computing the longitude of Ghazna relative to Baghdad, and the distance between those two cities and the *qibla* of Ghazna as well. In a much smaller treatise, the *Kitāb fī Tastīh al-suwar wa-tabtīh al-kuwar* [On The Projection of the Constellations and the Flattening of the Sphere],[47] he proposes an original mapping of a hemisphere which, when it was reinvented by G. B. Nicolosi in 1660, became known as the globular projection. He also describes a didactically interesting way of conceiving an earlier projection (known now as the azimuthal equidistant projection), invented (according to one account) by Khālid al-Marwarrūdhī.

Conclusions

Given the high level of Central Asian mathematics, it is unfortunate that European translators, who otherwise took so much from Arabic writings, knew so little of it. There are, of course, exceptions, such as the works of al-Khwārazmī. But, in general, Europe was ignorant of the major Central Asian mathematical works – and their authors – in the centuries when they could have had the most influence. Al-Birūnī, al-Kūhī, ᶜUmar Khayyām – none of them was known to European translators. One may easily suggest reasons why this was the case. Geographic separation, the particulars of the development of mathematics in the Muslim West (the primary site of transmission between Central Asia and Europe), the relative lateness of several of the major Central Asian writers and the interests and/or cultural

[45] See Lorch, 1987.
[46] See the review of this controversy in Hogendijk, 1984.
[47] See Berggren, 1982

level of the Latin translators: these are only a few of the reasons that come to mind as possible factors. One may debate their relative importance at some length, but what is beyond debate is that in the period covered by this volume, Central Asian mathematicians produced works which made their times one of the great ages of mathematical achievement.